NONFACTORIZATION IN AdS$_2$ QUANTUM GRAVITY

Ryan Alexander Marin

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Primary Advisor: Juan Maldacena
Secondary Advisor: Simone Giombi

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This work represents my own work in accordance with University regulations.

R. A. Mari

ABSTRACT

We study the phenomenon of nonfactorization in $1+1$ dimensional quantum gravity through the lens of Jackiw-Teitelboim (JT) gravity and its dual construction as an $\mathrm{SL}(2,\mathbb{R})^+$ BF theory. After constructing the classical and quantum Hilbert space, we derive the Schwarzian boundary dynamics, interpreting their residual symmetries as underpinning boundary entanglement and nonfactorization. From the Euclidean path integral perspective, we note how nonfactorization manifests as contributions from connected bulk moduli to the topological expansion. In Lorentzian signature, we analyze nonfactorization via entanglement in the Thermofield Double state $|\mathrm{TFD}(\beta)\rangle$ and discuss further the restrictions Lorentzian physics may impose on valid bulk moduli. Acknowledging these limitations, we outline a construction of the Saad–Shenker–Stanford [1] matrix integral which realizes Euclidean JT gravity as dual to a nonperturbative, ensemble-averaged boundary theory. To clarify the role of the ensemble in resolving the chaotic dynamics of JT gravity, we develop a fine-grained construction of the JT path integral which incorporates the full spectra of Schwarzian dynamics. Our discussion refines the multiple mechanisms of chaos in JT gravity, though we stress the need for further nonperturbative developments to clarify the relationships between JT gravity, ensemble averaging, and quantum complexity.

Acknowledgments

This thesis is the product of twenty-two years of curiosity. To this end, I dedicate my work to those who have provided me with unwavering encouragement along the way.

To Cole, Mom, and Dad: Your love and support for me since I could first form my own questions has been unwavering; I owe every word in this thesis to you three.

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We live on an island surrounded by a sea of ignorance. As our island of knowledge grows, so does the shore of our ignorance.

— John Archibald Wheeler

Contents

1 INTRODUCTION

From the genesis of their respective developments, quantum mechanics and general relativity have existed in a quiet tension. As a consequence of the former, theories of matter came to recognize that fundamental forces were not a continuous object, rather their interactions were better described by quanta: indivisible carriers of mass, charge, or spin. These theories of "quantum fields" describe particle interactions as localized, quantized excitations of these fields.

These fields, however, are subject to the Lorentz invariance postulated by general relativity. Just as Newtonian equations of motion are determined by their Euclidean symmetries, a quantum particle's behavior is understood by its transformation under the Lorentz group $SO(1,3)$, which manifests the symmetry[1] of Minkowski spacetime in general relativity.

Naturally, the mid-century goals of physics largely aimed at reconciling these two paradigms. Quantum Field Theory (QFT) was the treatment of these quantum fields in a relativistic sense, such that they respect the symmetries embedded in the structure of the particles and spacetime alike. However, in formulating these quantum field Lagrangians, it was noticed that many successful theories had a redundancy in their mathematical structure. For example, the theory of Electromagnetism — which characterizes the in-

[1] Including translations in space and time, the full symmetry group is actually ten-dimensional, given by the Poincaré group $\mathbb{R}^{3+1} \rtimes O(1,3)$

teractions of the Electric and Magnetic fields — can be more aptly described in a field-theoretic sense by a Lorentz-invariant four-vector potential, $\vec{\mathbf{A}}$. The QFT Lagrangian for this theory, whose equations of motion descend into Maxwell's equations, can be described by the Lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.1}$$

This Lagrangian contains a symmetry; a shift in the fields under which it is invariant. This is known as a local gauge symmetry [2]. In the Electromagnetic theory, it manifests as shifting $\vec{\mathbf{A}}$ by a total derivative, which shifts the Electromagnetic field by a local phase. One important consequence of local gauge symmetry is the correspondence with conserved internal symmetries of particles, such as charge or spin. If, for example, a charged scalar field interacts with the electromagnetic potential, the local gauge symmetry $\lambda(x)$ manifests as a $U(1)$ symmetry in the phase of the charge:

$$\mathcal{L}_{int} = -i(\partial_\mu \varphi^\star \varphi - \varphi^\star \partial_\mu \varphi)A^\mu \tag{1.2}$$

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x) \tag{1.3}$$

$$\varphi' = \varphi e^{i\lambda(x)} \tag{1.4}$$

These local gauge symmetries, or redundancies in the field description of the particle, give rise to the gauge bosons with the corresponding symmetry. In the case of the above $U(1)$ symmetry, the gauge boson is the photon (γ).

This is the framework on which the Standard Model is built. Further symmetries of the Weak/Strong interactions give subsequent $SU(2), SU(3)$ gauge symmetries, which give rise to the weak gauge bosons (W^+, W^-, Z^0) and (eight) gluons, respectively [3, 4, 5].

1.1 A Quantum Theory of Gravity

Given the successes of gauge theories in quantizing the non-gravitational forces, quantizing gravity was the natural next step for QFT to reconcile quantum mechanics and general relativity. Unfortunately, this has proven to be quite difficult. Along with the other gauge bosons, gravity was suspected to quantize into a spin-2 "graviton". However, for a quantum field description to be valid, it must be renormalizable: that is, it must not diverge at high energies. The perturbative methods in QFT, when applied to general relativity, proved gravity does not permit a perturbative, normalizable form [6].

The following decades were rich with theoretical insight: Inspiration from earlier attempts to incorporate gauge fields as extra compact[2] dimensions [7, 8] were reignited during the first Superstring revolution of the 1980s. Specific higher-dimensional manifolds were shown to, under compactification, reduce to renormalizable theories of gravity. These theories relied on a speculative supersymmetry (SUSY) between bosonic and fermionic matter, but they nonetheless have become standard tools of analysis due to their generality and elegance [5, 9]. These models are today understood as limiting cases of a more general, 11-dimensional M-theory [10] which defines the string theory land-

[2]In the language of manifolds, "compact" refers to periodicity in a dimension: circles are compact; lines are not.

scape. Perhaps the largest breakthrough, however, came in 1997, when the AdS/CFT correspondence was first realized by Juan Maldacena [11].

The correspondence can be understood as a dictionary between two theories: A "bulk" quantum gravity theory in a $d+1$-dimensional AdS spacetime, and a dual conformal field theory[3] living on the d dimensional boundary [12, 13]. A gravity state ϕ living in the bulk is characterized under this correspondence by the generating functional of correlation functions in the CFT:

$$\mathcal{Z}_{grav}[\phi_\partial(x)] = \langle e^{\int d^d x \phi_\partial(x) \mathcal{O}(x)} \rangle_{CFT} \tag{1.5}$$

More specifically, one can describe the asymptotic deformations of AdS space as a perturbative sum of CFT tensor expectation values:[4]

$$g_{\mu\nu}(z,x) \sim \frac{\eta_{\mu\nu}}{z^2} + z^{d-2} \langle T^{CFT}_{\mu\nu}(x) \rangle + ... \tag{1.6}$$

Theories deriving from this correspondence are referred to as "holographic", as the bulk physics can be entirely reconstructed from the boundary CFT information via this correspondence [14, 15]

The interest in AdS_2 gravity in particular is characterized by its anomalies. Two-dimensional gravity is unique in that the number of dynamical degrees of freedom reduces to zero, making the theory topological, thus dynamically trivial. Besides being particularly simple to work with, it also emerges in the near-extremal geometry of Reissner-

[3]A brief introduction to CFTs is provided in Appendix C

[4]Where z is the distance to the boundary, and $x^\mu = \{0, 1, ..., d-1\}$ parametrize the d-1 coordinates of the boundary.

Nordström black holes, where the event horizon can be understood as the conformal boundary of locally AdS_2 space [16]. In such a case, we also obtain a dynamical (dilaton) field φ corresponding to the transverse volume of the black hole, allowing for nontrivial dynamics: This description of gravity was first discovered in the 1980s, and is known as Jackiw-Teitelboim (JT) gravity [17, 18], which has an action given by:

$$S_{JT} = \frac{1}{16\pi G_2} \int_{\Sigma} d^2x \sqrt{-g} \left(\mathcal{R} - 2\Lambda_{\text{eff}}\right) \varphi \tag{1.7}$$

JT gravity is especially useful as a theoretical laboratory for quantum gravity research: It retains many of the topological features of generic two-dimensional gravity, yet it permits a rich-yet-solvable spectrum of dynamics which have continued to provide insight over the decades. The theory also benefits from having an alternative factorization in terms of "BF Theory", a topological gauge theory of gravity motivated from a more general formulation of General Relativity. In BF theory, one considers a scalar field "B" which (acts as a Lagrange multiplier), and a curvature two-form F derived from a gauge field A.

$$S_{BF} = \int_{\Sigma} \text{Tr(BF)}, \quad F = dA + A \wedge A \tag{1.8}$$

When A lives in an abelian gauge group, note that $F \propto F_{\mu\nu}F^{\mu\nu}$ as in equation (1.1). However, BF theory is more general than this — as we will see, a certain nonabelian gauge group $\mathcal{G} = \text{SL}(2, \mathbb{R})$ will provide us with a dual theory to JT gravity from which we may examine all of the defining properties of the theory.

In fact, one property of JT gravity in particular will prove exceptionally interesting.

In its Euclidean formulation, JT gravity possesses so-called “spacetime wormhole” solutions, which prevent the theory from factorizing. More specifically, one cannot in general describe states in JT gravity as dual to a unique tensor-product of states on the boundary. This may not seem significant at first, but it stands in violation to the traditional interpretation of AdS/CFT duality, where one expects bulk physics to have a unique, holographic, quantum-mechanical dual theory on the boundary.

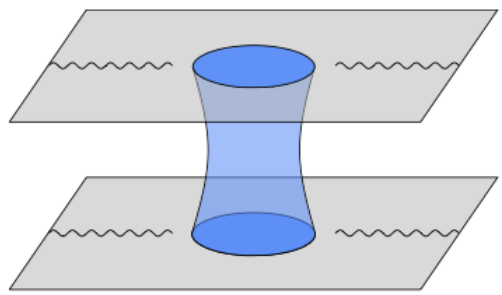

Figure 1: An illustration of a spacetime wormhole, connecting two (otherwise disconnected) regions of AdS spacetime. Illustration from [19].

For more complex bulk geometries, JT gravity forbids this: It is this puzzle of nonfactorization that has spurred much of the recent literature, forcing a confrontation with AdS/CFT.

Two main interpretations of this “nonfactorization puzzle” have arisen: The former, suggested by Harlow and Jafferis [20] in 2018, posits that nonfactorizing bulk geometries computed in the Euclidean signature do not prohibit a unique dual boundary theory. Their reasoning is that the Euclidean gravitational path integral (See Section 5.1) is not well-defined on a set of boundaries which do not connect a bulk: since nonfactorization arises only when considering each boundary as a unique Hilbert space, one cannot meaningfully describe such boundaries as unique in the first place. The Lorentzian formulation of JT gravity seems to support their conclusions, as many of the more complex bulk geometries are not well-defined as they are in the Euclidean signature.

The alternative perspective has been much more influential, however, due to its remarkable conclusions. In the same year, Saad, Shenker, and Stanford[5] proved in their work "JT Gravity as a Matrix Integral" [1] that the full Euclidean expansion of JT gravity can be expressed non-perturbatively as dual to a double-scaled random matrix integral. That is, their work proved a duality between Euclidean JT gravity and a boundary theory which is not described by a unique Hamiltonian, but by an "ensemble" of Hamiltonians, where a given state's eigenvalue distribution depends on a statistical distribution of boundary theories, rather than a particular one:

$$\mathcal{Z} = \int dH\, e^{-L\mathrm{Tr}(V(H))} \tag{1.9}$$

Where the full partition function then requires an integral over *all* Hamiltonians, where the given probability of a particular draw is determined by a potential function $V(H)$. This has the consequence that partition functions cannot be computed exactly; the partition function for a connected manifold with (n) boundaries is then an expectation value.

$$\mathcal{Z}_n(\beta_1, ..., \beta_n)\rangle_{\mathrm{conn}} \Rightarrow \langle \mathcal{Z}_n(\beta_1, ..., \beta_n)\rangle_{\mathrm{conn}} \tag{1.10}$$

[5]Since this trio appears constantly throughout this work, we herein refer to them (and their seminal work) as "SSS".

Explicit matrix models — such as the Sachdev-Ye-Kitaev (SYK) theory [21, 22] — which share the properties of the random matrix dual in Saad Shenker and Stanford have been discussed, however, a full interpretation of the meaning and depth of the random matrix dual remains to be fully understood.

Due to the significance of nonfactorization and its divergent interpretations for holography and quantum gravity, this thesis seeks to understand the rationale and potential reconciliation of these divergent opinions, ideally culminating in a unified understanding of nonfactorization and ensemble averages, and how they reconcile with the search for a quantum theory of gravity.

1.2 Outline

To reach a discussion on the above issues, we must first begin with the development of the underlying theories. In chapter 2, we begin by tracing the lineage of JT gravity, deriving it from the familiar Einstein-Hilbert action. We also reformulate the theory in the language of BF theory, introducing mathematical machinery as we go. In chapter 3, we introduce BF theory in more generality, striving for pedagogical clarity toward its classical observables and symmetries.

In chapter 4, we quantize the theory — this is a necessary step to understand the microstate structure of the theory, especially when considering the full space of boundary dynamics. This will require an examination of gauge fixing to determine the quantum

Hilbert space of the theory, especially the states on the boundary. We also introduce the Wick-rotated, Euclidean version of the theory, providing a more sophisticated analysis of the full structure of the $SL(2, \mathbb{R})$ theory. Armed now with the necessary formalism, in chapter 5 we dive into the factorization puzzle, exploring it from the perspective of the Euclidean gravitational path integral. This naturally leads to a discussion on wormholes and entanglement, considering nonfactorization from a Lorentzian perspective on Thermofield Double (TFD) states.

In chapter 6, we explore the multiple resolutions to nonfactorization in greater detail. We begin with an overview of the Harlow-Jafferis perspective, yet spend a majority of the section constructing the Saad-Shenker-Stanford (SSS) ensemble dual, weaving together random matrix theory, moduli spaces, and topological expansion in the process. In chapter 7, we expand on the SSS construction, reformulating the procedure such that the "fine-grained" boundary structures are preserved. We analyze the early-time behavior of the leading contribution Spectral Form Factor (SFF), utilizing it as a heuristic, heavily-qualitative tool to probe the potential separate origins of chaos in JT gravity. The section then concludes with a discussion on the multifaceted role of chaos evident from the fine-grained prescription, as well as its suggestive connections to other open questions in quantum complexity theory. Chapter 8 provides a retrospective summary of the work, as well as some concluding remarks.

2 Classical AdS_2 Gravity

To understand and motivate the study of JT gravity, it is vital to understand where it arises in nature. Unlike most AdS_n theories of gravity which are studied for their mathematical simplicity, JT gravity is unique in that it arises naturally at the horizons of charged, non-rotating black holes known as Reissner-Nordström black holes. In the standard formulation of Einstein-Hilbert gravity, the lack of dynamical degrees of freedom in the metric give an action that is diffeomorphism-invariant, proportional to the Euler characteristic of the spacetime:

$$S_{EH} = \int_M d^2x \sqrt{-g}\mathcal{R} = 4\pi\chi(M) \tag{2.1}$$

In the 1980s, it was discovered that 2D gravity becomes much more dynamical–and interesting–when an additional background ”dilaton” field was introduced [17]. This field couples to the Ricci curvature of the spacetime, breaking the diffeomorphism invariance and leading to nontrivial equations of motion. While the original motivation for the dilaton was string-theoretic, it manifests in naturally as the “volume” of the cross-sectional element in a compactified spacetime. As we shall see, in the case of a black hole, one can consider the action restricted only to the $\{t, r\}$ coordinates, contracting the transverse S^2 to a scalar-valued dilaton field Φ.

2.1 EINSTEIN-HILBERT GRAVITY

We begin this story with the standard Einstein-Hilbert action on a flat, $\mathbb{R}^{3+1}$ Lorentzian spacetime with a background Electromagnetic field:[6]

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{-g_4}(R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}) \tag{2.2}$$

Since we will be considering the reduction of this action around a (near) extremal Reissner-Nordström black hole, we can assume that the solution will contain S^2 symmetry, and hence we rewrite this action under a spherically-symmetric ansatz:

$$ds^2 = g^{(2)}_{\mu\nu}(x)dx^\mu dx^\nu + \Phi(x)^2 d\Omega_2^2 \tag{2.3}$$

$$F = Q\sin(\theta)d\theta \wedge d\phi \tag{2.4}$$

Where $\Phi(x)$ encodes the radius, giving a constant value on $d\Omega_2^2$, and μ, ν are indices $\{t, r\}$ labeling the non-compact coordinates. Denoting $\mathcal{R}$ to be the two-dimensional Ricci curvature, we rewrite the Einstein Hilbert action in terms of the $SO(3)$-symmetric variables

$$\begin{gathered} R = \mathcal{R} - \frac{2}{\Phi}\nabla^2\Phi - \frac{2}{\Phi^2}(\nabla\Phi)^2 + \frac{2}{\Phi^2} \\ g_4 = g\Phi^4\sin^2(\theta) \end{gathered} \tag{2.5}$$

We remove boundary terms, and are left with the following generic solution [23]:

$$S_{2D} = \frac{1}{16\pi G}\int d^2x\sqrt{-g}[\Phi^2\mathcal{R} + 2(\nabla\Phi)^2 + 2] \tag{2.6}$$

[6] G will herein be the Gravitational coupling constant in four dimensions.

Which we will insert into the Reissner-Nordström solution. This is the most general form which the Einstein-Hilbert action in a massless background can take in two dimensions.

2.1.1 Robinson–Bertotti Geometry and the AdS$_2$ Throat

The Reissner-Nordström [RN] solution [24][25] represents a static black hole with mass and charge $\{M, Q\}$. As visible from the solutions for $r_{\pm}$, this solution is unique from its non-charged counterpart in that it has two event horizons. However, in the limit where a RN black hole accumulates enough charge, it approaches a distinct geometric change in its geometry: at $Q \lesssim M\sqrt{G}$, the gap between the horizons vanishes, and the temperature approaches zero. In this limit, the RN black hole develops an AdS$_2$ "throat", and the geometry factorizes into $AdS_2 \times S^2$: This is known as the Robinson-Bertotti (RB) geometry [26, 27].

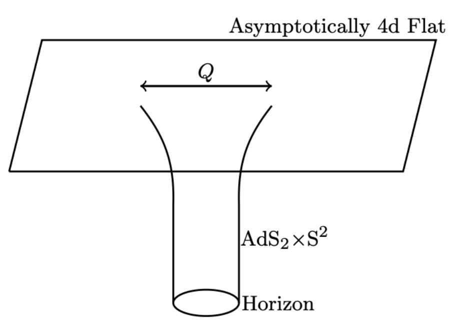

Figure 2: An illustration of the "splitting" of the Robinson-Bertotti geometry. (Photo from [23])

This near-extremal, low-energy regime is of particular interest to JT gravity, as the near-horizon geometry here develops an AdS$_2$ "throat" on the order of

$$\ell_{\text{throat}} \sim \log\left(\frac{1}{T}\right) = \log\left(\frac{4\pi r_+^2}{r_+ - r_-}\right) \tag{2.7}$$

Where T is the Hawking temperature of the black hole [28, 29]. We hence derive the conditions appropriate for the application of JT gravity, deriving the dilaton as a byproduct of the spherical area of this throat. Consider the Reissner-Nordström metric:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2 \tag{2.8}$$

This metric has coordinate singularities at the inner and outer radii, given by

$$r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2} \tag{2.9}$$

Which, as alluded to above, are equivalent at extremality. Taking this limit geometrically, we take the following limit on r, using ρ as a dimensionless parameter and taking

$$r = r_+ + \epsilon\rho \qquad \epsilon \to 0 \tag{2.10}$$

Which, substituted into the above metric yields:

$$f(r) = \frac{\epsilon^2\rho^2}{Q^2G} + \mathcal{O}(\epsilon^3) \qquad d\rho = \frac{1}{\epsilon}dr \tag{2.11}$$

$$ds^2 = -\frac{\epsilon^2\rho^2}{Q^2G}dt^2 + \frac{Q^2G}{\rho^2}d\rho^2 + Q^2Gd\Omega_2^2 \tag{2.12}$$

Rescaling time such that $\tau = \frac{Q\sqrt{G}}{\epsilon}t$, we are left with

$$ds^2 = Q^2G(-\rho^2dt^2 + \frac{d\rho^2}{\rho^2} + d\Omega_2^2) \tag{2.13}$$

Which gives precisely the Robinson-Bertotti geometry, $AdS_2 \times S^2$.

2.1.2 Dilaton from Entropy: Jackiw-Teitelboim Gravity

We now wish to marry these two pieces of insight into obtain the expression for JT gravity. To connect this to the 2 dimensional action (2.2), recall we defined $\Phi = r$, and thus in the near-extremal limit:

$$\Phi^2 = r^2 = Q^2 G + 2\epsilon\rho Q\sqrt{G} + \mathcal{O}(\epsilon^2) \tag{2.14}$$

$$(\nabla\Phi)^2 = 0 + \mathcal{O}(\epsilon^2) \tag{2.15}$$

where $\epsilon \ll 1$ parametrizes deviations from extremality. The vanishing of $(\nabla\Phi)^2$ at leading order is a consequence of the Robinson-Bertotti (RB) geometry near the horizon [26, 27]. In this limit, The RB geometry $AdS_2 \times S^2$ admits only modes that are symmetric on S^2 in the low-energy limit; that is, off-diagonal excitations (e.g., $g_{\mu\theta}$) would diverge due to the infinite throat and are thus excluded [11].

In line with the literature, we also assign a new variable, φ, to account for the small near-extremal deviation in Φ^2:

$$\Phi^2 \approx Q^2 G + \varphi(x) \tag{2.16}$$

Which encodes the transverse area of S^2 in the near-horizon limit. With these conditions, we substitute back into the action (2.4):

$$S_{2D} = \frac{1}{16\pi G}\int d^2x\sqrt{-g}[(Q^2 G + \varphi(x))\mathcal{R} + 2] \tag{2.17}$$

Where our AdS_2 has characteristic scale $\Lambda = -2/l^2$, which rescales to $\Lambda_{\text{eff}} = -1/G_4^2$ By the Gauss-Bonnet theorem, the term $\propto \sqrt{-g}$ is topological and does not contribute to the action. Rescaling units, we are left with

$$S_{JT} = \frac{1}{16\pi G_2}\int d^2x\sqrt{-g}(\mathcal{R} - 2\Lambda_{\text{eff}})\varphi \tag{2.18}$$

Which gives the JT Gravity action in the bulk AdS$_2$. By varying φ, we see that the action enforces a constant negative curvature $\mathcal{R} = 2\Lambda_{\text{eff}}$. To get the equation of motion for φ, we vary (2.18) in $\delta g_{\mu\nu}$:

$$\delta S_{\text{bulk}} = \frac{1}{16\pi G}\int d^2x\,\sqrt{-g}\left[\delta(\sqrt{-g})\,\varphi(\mathcal{R}+2) + \varphi\,\delta\mathcal{R}\right], \tag{2.19}$$

$$\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}, \quad \delta\mathcal{R} = \left(\mathcal{R}_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla^2\right)\delta g^{\mu\nu}, \tag{2.20}$$

$$\delta S_{\text{bulk}} = \frac{1}{16\pi G}\int d^2x\,\sqrt{-g}\left[\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\nabla^2\varphi - g_{\mu\nu}\varphi\right]\delta g^{\mu\nu}, \tag{2.21}$$

$$\nabla_\mu\nabla_\nu\varphi - g_{\mu\nu}\nabla^2\varphi + g_{\mu\nu}\varphi = 0 \tag{2.22}$$

Which, after contracting with $g^{\mu\nu}$, gives the condition on φ:

$$\nabla_\mu\nabla_\nu\varphi = g_{\mu\nu}\varphi \tag{2.23}$$

This is a condition which aligns with intuition: since JT gravity has no dynamical degrees of freedom, a non-constant φ would break SL$(2,\mathbb{R})$ invariance. Thus φ is dynamical only on the boundary, and has no freedom in the bulk [30]. The discussion of the dynamics of φ on the boundary will be left for section 4.

2.2 Boundary Schwarzian Action

Omitted in the above derivation is the recognition of the fact that, as $r \to \infty$, we require a "cutoff" between the $AdS_2 \times S^2$ throat and the asymptotically $\mathbb{R}^{3+1}$ Minkowski geometry at infinity. To address this, we consider first the full JT action in the Euclidean signature, defined on an AdS_2 manifold M with conformal boundary ∂M.[7]

$$S_{JT+\partial}[g,\Phi] = -S_0\chi - \frac{1}{16\pi G_2}\int_M \sqrt{g}\Phi(\mathcal{R}+2) - \frac{1}{8\pi G_2}\int_{\partial M}\sqrt{h}\Phi(K-1) \tag{2.24}$$

Where χ is the aforementioned topological component, and h, K are the metric and curvature localized to the boundary. According to the equations of motion from S_{JT}, we have that $\mathcal{R} = -2\Lambda_{\text{eff}}$, and thus this term does not contribute to the dynamics. We are interested in the dynamics on the boundary; the bulk metric is fixed, and we allow the boundary to fluctuate. Recall that the asymptotic RB geometry was given to be

$$ds^2 \to Q^2 G\left(-\rho^2 dt^2 + \frac{d\rho^2}{\rho^2} + d\Omega_2^2\right), \tag{2.25}$$

which defines the throat region of the black hole geometry. The AdS_2 factor governs the radial-time sector, and is precisely the geometric background on which JT gravity is defined after dimensional reduction. To evaluate the boundary action for JT gravity, recall that $\rho \to \infty$ corresponds to the conformal boundary infinitely far from the horizon. This metric is only asymptotically exact, hence we fix a cutoff distance $\rho = \rho_c$ where the dilaton has fixed value φ_c.

Consider the boundary in (t,ρ) coordinates parameterized by $x^\mu(u) = \{t(u), \rho(u)\}$ for $u \in \mathbb{R}$. The induced boundary metric is then

$$ds^2_{\text{bdry}} = C^2\left(\rho(u)^2\dot{t}^2 - \frac{\dot{\rho}^2}{\rho(u)^2}\right) du^2 \tag{2.26}$$

[7]This includes the Gibbons-Hawking-York boundary term. For more discussion on the origin of this term, see [23].

Where we have omitted constants and the transverse $d\Omega_2^2$ for clarity. Fixing the length of the conformal boundary as

$$ds^2_{\text{bdry}} = -\frac{1}{\varepsilon^2}du^2 \Rightarrow \rho^2 \dot{t}^2 - \frac{\dot{\rho}^2}{\rho^2} = -\frac{1}{C^2\varepsilon^2} \tag{2.27}$$

Which gives an expression from which we may derive the JT gravity action. Note that these boundary reparameterizations have no effect on the topological bulk action. To calculate δS_{JT}, expanding $\rho(u)$ as $\rho(u) = \frac{1}{\epsilon} + \epsilon\rho_1(u) + \mathcal{O}(\epsilon^2)$ one has may compute

$$\delta S_{JT} = \frac{1}{8\pi G}\int_{\partial M} \Phi\sqrt{h}(\delta K - 1) \tag{2.28}$$

From fluctuations in ρ. Note that $\Phi(u) = \Phi_r/\epsilon$ is constant at the boundary by design, and $\sqrt{h} \propto \frac{C}{\epsilon}$. Expansion in ϵ on K yields:[8]

$$K = 1 + \epsilon^2\left[\frac{f'''(u)}{f'(u)} - \frac{3}{2}\left(\frac{f''(u)}{f'(u)}\right)\right] + \mathcal{O}(\epsilon^4) \tag{2.29}$$

$$K :\approx 1 + \epsilon^2\{f(u), u\} \tag{2.30}$$

From which we derive the Schwarzian action:

$$S[f] = -\frac{\Phi_r}{8\pi G}\int_{\partial M} du\{f(u), u\} \tag{2.31}$$

Where the bracket $\{f(u), u\}$ is known as the Schwarzian derivative. An interesting artifact of this action is that it is precisely equivalent to the space of Diff$(S^1)/\mathrm{SL}(2,\mathbb{R})$; in other words, excited modes in the boundary theory correspond to boundary fluctuations

[8] see [23] for the detailed algebra.

not contained in $\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{ISO}(\mathrm{AdS}_2)$. This can be realized explicitly by examining the vanishing locus of Schwarzian derivatives:

$$\{f(u), u\} = 0 \quad \Leftrightarrow \quad f(u) = \frac{au+b}{cu+d}, \quad ad - bc = 1 \tag{2.32}$$

$$\{f(u), u\} = \{g(u), u\} \Leftrightarrow F \sim G \text{ under } \mathrm{PSL}(2,\mathbb{R}) \tag{2.33}$$

Hence Schwarzian boundary states form an infinite-dimensional basis of non isometric, monotonic diffeomorphisms of S^1:

$$|f\rangle \in \mathrm{Diff}^+(S^1)/\mathrm{SL}(2,\mathbb{R}) \tag{2.34}$$

These $\mathrm{SL}(2,\mathbb{R})$ symmetries which leave the boundary action invariant are equivalent to the isometry group of AdS_2 for a reason: the invariant transformations of the bulk should hence leave observables invariant, and therefore correspond to a gauge group on the boundary. As we will see in section 4, these Schwarzian modes are precisely those parameterizations of the boundary which cannot be "gauged away" with bulk gauge symmetry.

2.3 Topological Formulation: SL(2, $\mathbb{R}$) BF Theory

The above calculations suggest that JT gravity has an internal $\mathrm{SL}(2,\mathbb{R})$ symmetry. In the same fashion that Kaluza and Klein understood the $\mathrm{U}(1)$ gauge symmetry of Electromagnetism as descending from a fifth, compactified dimension in 4D Einstein gravity [7, 8], it has been shown that the $\mathrm{SL}(2,\mathbb{R})$ symmetry can be treated in the same fashion. To formalize this, we will consider the "Zweibein" formulation of JT gravity, arriving at an equivalent topological gauge theory.

2.3.1 The Zweibein formalism

The Zweibein (German for two-legged) formalism is a geometric reframing of general relativity which substitutes the metric tensor $g_{\mu\nu}$ for a local orthonormal frame. For any point p in a Lorentzian $1+1$ dimensional manifold M, the tangent/cotangent vectors are given by:

$$e_\mu = \partial_\mu \quad e^\mu = dx^\mu \quad \mu \in \{t, r\} \tag{2.35}$$

Where greek indices indicate that these objects span the (co)vector space on the (potentially curved) background spacetime. If we want to take advantage of the fact that M is smooth, we use latin indices to express this fact ($\langle e_a, e_b \rangle = \eta_{ab}$). Just as in the typical case of locally-valued one forms in general relativity. The purpose of this is to connect the local frame to the global one; the Zweibein is the matrix e^a_μ with relations:

$$e_\mu = e^a_\mu e_a \quad e_a = e^\mu_a e_\mu \quad \text{Local} \leftrightarrow \text{Global Coordinates} \tag{2.36}$$

$$e^\mu_a e^a_\nu = \delta^\mu_\nu \quad e^a_\mu e^\mu_b = \delta^a_b \quad \text{Orthonormality} \tag{2.37}$$

$$g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \quad \text{Local} \rightarrow \text{Global} \tag{2.38}$$

$$\eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b \quad \text{Global} \rightarrow \text{Local} \tag{2.39}$$

$$\Lambda^{a'}_a \Lambda^b_{b'} \eta_{ab} = \eta_{a'b'} \quad \text{Lorentz Transformations} \tag{2.40}$$

This gives us a way of representing any vector in the tangent space, where $A \in T_p(M)$

is written as $A = A^{\mu} e_{\mu}$. There is one final element in this formalism, however, that is analogous to the Christoffel symbols $\Gamma^{\rho}_{\mu\nu}$ in general relativity, and that is the so-called "spin connection", $\omega = \omega_{\mu} dx^{\mu}$. The spin connection is a one-form which connects local Lorentz frames $\{e_a\}$ at different points in the global spacetime, and serves the analogous role of the connection coefficients $\Gamma^{\rho}_{\mu\nu}$ in classical general relativity.

Recall from general relativity that the derivative of a vector field does not transform naturally under Lorentz transformations unless we add connection coefficients. That is,

$$\nabla_{\mu} X^{\nu} = \partial_{\mu} X^{\nu} + \Gamma^{\nu}_{\rho\mu} X^{\rho} \tag{2.41}$$

$$\Rightarrow \Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu} \right) \tag{2.42}$$

The equivalent expression in the Zweibein formalism is:

$$\nabla_{\mu} X^{a} = \partial_{\mu} X^{a} + \omega^{a}_{\mu c} X^{c} \tag{2.43}$$

$$\omega^{ab}_{\mu} = e^{a}_{\nu} \Gamma^{\nu}_{\rho\mu} e^{\rho b} + e^{a}_{\nu} \partial_{\mu} e^{\nu b} \tag{2.44}$$

$$\Rightarrow \omega_{\mu} = \frac{1}{2} \epsilon^{ab} \, e^{\nu}_{a} \left(\partial_{\mu} e_{\nu b} - \partial_{\nu} e_{\mu b} + e^{\rho}_{b} \, e_{\mu c} \, \partial_{\nu} e^{c}_{\rho} \right) \tag{2.45}$$

Where ω_{μ} has just one nontrivial term, $\omega^{01} := \omega$, as the diagonal terms vanish due to antisymmetry. In the case of $\Gamma^{\rho}_{\mu\nu}$, we derive their unique form from the zero-torsion condition. Similarly, the First Cartan equation gives the torsion in terms of the Zweibein, allowing for a unique solution of ω:

$$T^{a} = de^{a} + \omega \wedge \epsilon^{a}_{b} e^{b} = 0 \tag{2.46}$$

The pair $\{e_a^\mu, \omega\}$ formally define the Zweibein.[9]

2.3.2 From the Zweibein to BF Theory

Recall the JT gravity action given by (2.18):[10]

$$S_{JT} = \frac{1}{16\pi G_2}\int d^2x(\mathcal{R} - 2\Lambda_{\text{eff}})\varphi \tag{2.47}$$

We wish to construct this action in the BF theory formalism. To do so, consider a two-dimensional Lorentzian manifold M which has a gauge group $SO(2,1) \cong SL(2,\mathbb{R})$. This group is exactly the group of symmetries which act transitively on the AdS_2 spacetime, and can hence be understood geometrically as the hyperboloid:

$$\eta^{ab}z_a z_b - \ell^2 = 0 \tag{2.48}$$

Where ℓ is the characteristic AdS scale, and $T_z(M) = \text{span}(\{e^a\})$ In the Zweibein formalism, this symmetry will correspond to the gauge symmetry of fields living on M. We will ultimately want an action which resembles S_{JT}, so we start with a 0-form B and a one-form A defined as follows:[11]

$$A = e^a_\mu \tau_a dx^\mu \quad B = B^a\tau_a \tag{2.49}$$

Where τ_a are the elements of the Lie algebra $\mathfrak{so}(1,2)$, and thus have the commutation relations:

$$[\tau_a, \tau_b] = -\epsilon_{abc}\tau^c \quad \text{s.t.} \quad a,b,c \in \{0,1,2\} \tag{2.50}$$

[9] It is common in the literature to denote $e^2_\mu = \omega_\mu$.

[10] Where we set $\Lambda_{\text{eff}} = 1$ for clarity

[11] Where we use the shorthand $e^2_\mu = \omega_\mu$

These are the infinitesimal generators of the isometry group $SO(2,1)$, and hence can be raised or lowered with the metric $\eta_{ab} = \text{diag}(1,-1,-1)$. As in the case of Maxwell's equations, we can compute the curvature two-form F from the gauge group by taking the curvature of the gauge field,

$$F_{\mu\nu} = F^a_{\mu\nu}\tau^a = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = dA + A \wedge A \tag{2.51}$$

Plugging in our expression (2.50) yields

$$F^a_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu - \epsilon^a_{bc} e^b_\mu e^c_\nu \tag{2.52}$$

To get to JT gravity, we ought to remember that the action is (on the bulk) topological, and thus should not have any dynamical degrees of freedom. Just as the equation of motion from JT gravity gives us the Liouville equation $\mathcal{R} - \Lambda = 0$, we would like a similar equation on F (recall $F^2_{\mu\nu} = \mathcal{R}_{\mu\nu}$). This is where the B field becomes useful as a Lagrange multiplier on F such that the action restricts to flat connections on M. Contracting F with B and integrating over M yields:

$$S_{BF} = \int B_a F^a_{\mu\nu} \epsilon^{\mu\nu} d^2x = \int_M \text{tr}(B \wedge F) \tag{2.53}$$

Which is the desired form of our action.

2.3.3 Recovering the S_{JT} from S_{BF}

The previous derivation may leave some readers unconvinced. To remedy this, we can check this action is equivalent to (2.18) by substituting in the relevant expressions and expanding out [31]. Recall the definition of A, B given above in terms of the SO$(2,1)$ symmetry generators, where we expand and add in constants for clarity:

$$A = \sqrt{\Lambda_{\text{eff}}} e^a \tau_a + \omega(x)\tau_2, \quad B = B^a \tau_a + B^2 \tau_2, \quad a \in \{0,1\} \tag{2.54}$$

We can plug these into (2.53), obtaining

$$S_{BF} = \int_M \sqrt{\Lambda_{\text{eff}}}[B^1(de^1 + \omega \wedge e^0) + B^0(de^0 - \omega \wedge e^1)] - B^2(d\omega + \Lambda_{\text{eff}} e^0 \wedge e^1) \tag{2.55}$$

Varying with respect to B^a yields

$$de^a + \omega^a_b \wedge e^b = 0 \tag{2.56}$$

Which is precisely equation (2.46), ensuring torsion vanishes. Furthermore, from (2.53), we have the equation of motion by varying the scalar B that $F = 0$. Together, these reduce (2.55) to the following form:

$$S_{BF} = \int_M e^0 \wedge e^1 (d\omega + \Lambda_{\text{eff}} e^0 \wedge e^1) \tag{2.57}$$

This form of (2.53) is deceptively close to the original JT action we derived in the beginning of section 2. $e^0 \wedge e^1 = e^0_\mu e^1_\nu dx^\mu \wedge dx^\nu = \sqrt{g}\ d^2x$, and $d\omega$ is related to the Ricci curvature via the Cartan Structure Equation [32, 33]:

$$d\omega = \frac{\mathcal{R}}{2} e^0 \wedge e^1 \tag{2.58}$$

Which reduces equation (2.57) to the form:

$$S_{BF} = \frac{1}{2}\int_M d^2x \sqrt{g} B^0 (\mathcal{R} + 2\Lambda_{\text{eff}}) \tag{2.59}$$

Which matches the original JT gravity action precisely.

2.3.4 ZWEIBEIN EQUATIONS OF MOTION

Before we begin with a deep analysis of JT gravity, it will be instructive to see the correspondence between objects, symmetries, and equations of motion in both theories. Consider the generic form for S_{BF} given by (2.53); varying B gives us:

$$\delta_B S_{BF} = \int_M \mathrm{tr}(\delta B \wedge F) \tag{2.60}$$

$$F = 0 \tag{2.61}$$

Which recovers the flatness condition, corresponding to the Liouville equation in JT gravity ($\mathcal{R} = 2\Lambda_{\text{eff}}$). Varying in F gives

$$\delta_F S_{BF} = \int_M \mathrm{tr}(B \wedge \delta F) \tag{2.62}$$

$$\delta_F S_{BF} = \int_M \mathrm{tr}(B \wedge (d\delta A + [A, \delta A])) \tag{2.63}$$

Which we integrate by parts on the first term to recover

$$\delta_F S_{BF} = \int_M \mathrm{tr}((dB + [A, B]) \wedge \delta A) \tag{2.64}$$

$$\mathcal{D}B = dB + [A, B] = 0 \tag{2.65}$$

Which matches the intuition that B functions as φ does in JT gravity, and thus $\mathcal{D}B = \nabla\varphi = 0$. We also have a correspondence in the symmetries: the gauge symmetry in A is given by elements of $SO(2,1)$, where $SO(2,1) \cong SL(2,\mathbb{R}) \cong \mathrm{ISO}(AdS_2)$ corresponds to the isometry group of the bulk in JT gravity. In essence, we have substituted a geometric symmetry for a gauge symmetry, which will prove to be a powerful tool in the following chapters.

3 BF Theory

Now that we have derived BF Theory as a natural reformulation of JT gravity, it will be productive to analyze its structure more deeply. For the sake of clarity, we will restrict our analysis here to the case of $SL(2,\mathbb{R})$ BF Theory, though the case of a general compact $\mathcal{G}$ follows similarly from our treatment of $SU(2)$ in Chapter 4. Our analysis will begin with a structured overview of the general equations of motion, constraints, and observables which arise in this theory. We will then use techniques of partial gauge fixing to analyze the phase space of the theory, analogous to the Coulomb or Temporal gauge in QFT [34]. Finally, we will perform a symplectic reduction on the phase space to determine the true gauge-invariant quantities which determine the physical phase space of the theory.

In it's differential form, BF theory is commonly written as

$$S_{BF} = \int_M B^i F^j \eta_{ij} \tag{3.1}$$

Where $\eta_{ij} = \text{diag}(-1,1,1)$, corresponding to the Lie algebra $\mathfrak{so}(1,2)$ of $SL(2,\mathbb{R})$[12]. As mentioned above, we can write the equations of motion in differential form as:

$$F^i := dA^i + \frac{1}{2} f^i_{jk} A^j A^k = 0 \tag{3.2}$$

$$\mathcal{D}_A B = dB + f^i_{jk} A^j B^k = 0 \tag{3.3}$$

[12]This definition follows from the fact that $\eta_{ij} = \frac{1}{2} f^l_{ik} f^k_{jl}$. See [35].

Where f^i_{jk} are the structure constants of $\mathfrak{so}(1,2)$. Note that in comparison to the Zweibein formulation, A^0, A^1 correspond to the Zweibein components e^i, and $A^2 = \omega_{01}$. This lie algebra has three generators: spatial translations, temporal translations, and dilations (Lorentz boosts). These will herein be notated J_0, J_1, and J_2 respectively. Their commutation relations are:

$$[J_i, J_j] = f^k_{ij} J_k \qquad f^2_{01} = f^1_{20} = 1; f^0_{12} = -1 \tag{3.4}$$

Which generate the $\mathfrak{so}(1,2)$ Lie algebra. This gives us the mathematical support to determine how these equations transform under a gauge transformation. Let $\lambda = \lambda^i J_i \in \mathfrak{so}(2,1)$ be an infinitesimal transformation. Just as how in a Maxwellian gauge transformation we have that $A'_\mu = A_\mu + \partial_\mu \lambda$, for the connection A above we have

$$\delta_\lambda A = \partial \lambda + [A, \lambda] \tag{3.5}$$

$$A' = gAg^{-1} + gdg^{-1} \tag{3.6}$$

For $g = e^\lambda \in SO(2,1)$. The 0-form B transforms as a matter field under gauge transformation under $ad(G)$:

$$B = gBg^{-1} \tag{3.7}$$

Such that $\langle B, F \rangle$ is invariant. Note that, per the equations of motion, we require $F = 0$, and thus A must be flat. This is equivalent modulo gauge transformation to the statement that $A = dgd^{-1}$ for some $g(x) \in G$ defined on M.

3.1 Observables and Invariants

Now that we have defined how BF theory transforms within its gauge group, we wish to compute the space of invariants of this theory. This is a necessary step for understanding the states of the theory, since gauge-invariance applies intrinsically to all observables. To do this, let us first examine the facts: we know that A is a flat connection on M, and furthermore we know that $\mathcal{D}B = 0$, or that B is covariant constant on M. For the needs of our application of BF theory, we can assume that $M = \Sigma \times \mathbb{R}$, where $\mathbb{R}$ parameterizes time, and Σ is a one-dimensional spatial manifold. By the equations of motion, to find invariant functionals of A, B, we must consider quantities which are invariant under the gauge group G; that is,

$$\{\mathcal{O}, G\} = 0 \tag{3.8}$$

By our foliation of M as $1+1$, we can rewrite the action (3.1) as:

$$S_{BF} = \int_{\mathbb{R}} dt \int_{\Sigma} dx (B_i \dot{\omega}^i_x + \omega^i_t \mathcal{D} B_i) \tag{3.9}$$

$$A = e^a J_a + \omega J_2, \quad B = \eta^a J_a + \omega J_2, \quad a \in \{0, 1\} \tag{3.10}$$

Where we can now express $\mathcal{D}B^i = 0$ in terms of the Lie algerba basis:

$$g_0 = \partial_0 B^0 - (\omega^1 B^2 - \omega^2 B^1) = 0 \tag{3.11}$$

$$g_1 = \partial_1 B^1 + \omega^2 B^0 - \omega^0 B^2 = 0 \tag{3.12}$$

$$g_2 = \partial_2 B^2 - \omega^0 B^1 - \omega^1 B^0 = 0 \tag{3.13}$$

Giving us three constraints on ω_x, which naively permits zero degrees of freedom on any given time slice, implying A_x collapses in the phase space. However, this is only true locally: If Σ has a nontrivial topology, then the global quantity

$$T = \mathrm{Tr}(\mathcal{P}e^{\oint_\Sigma A^i J_i}) = \mathrm{Tr}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\mathcal{P}\oint_{\Sigma_1} A \oint_{\Sigma_2} A \cdots \oint_{\Sigma_{n-1}} A\right) \in G \tag{3.14}$$

Which will correspond to the Wilson loop operator under quantization. This will be our first invariant, $\mathcal{O}_A$. Note that, by this definition, any gauge-invariant function on A in Σ must be a class function of the non-contractible cycles in A:

$$\Psi(U_1, U_2, \cdots, U_n) \tag{3.15}$$

Where U_k specifies the holonomy for each generator of $\pi_1(\Sigma)$ [36]. Furthermore, it was seen in equation (3.7) that B transforms under conjugation, so any B-dependent invariant observable must also be invariant under conjugation. In other words, the observable $\mathcal{O}_B$ must be a class function of g. The total class of these is given by

$$\mathrm{Tr}(P[B]) = \mathrm{Tr}\left(\sum_{n=0}^{N} a_n B^n\right) \tag{3.16}$$

However, $\mathrm{Tr}(B) = 0$ identically, so $\mathcal{O}_B = \mathrm{Tr}(B^2)$ is the simplest gauge-invariant quantity. This is known as the *casimir* of the representation of G, and it is the defining gauge-invariant quantity for B. Note that these conditions restrict (3.16) to be dependent only on the even powers of $\mathrm{Tr}(B^k)$, and thus any polynomial can be constructed out of $\mathrm{Tr}(B^2)$ as an observable.

3.2 SYMPLECTIC STRUCTURE AND CLASSICAL CONSTRAINT SURFACE

To continue with our analysis of the phase space of BF theory, it will be fruitful to consider the symplectic structure of the theory. The symplectic structure, denoted ω, is a two-form defined on the phase space of solutions which generates the Hamiltonian flow; it will provide us with the framework needed to rigorously define the phase space of the theory. For a review of the symplectic formulation of classical mechanics, see [37]

From the action (3.10), we can immediately recover the BF lagrangian as

$$\mathcal{L}_{BF} = \int dx \left(B^i \partial_t A^i_x + A^i_t D_x B^i \right) \tag{3.17}$$

In terms of the fields (A, B). Using Hamilton's equations, one can further define the conjugate momenta π of A^i_μ as

$$\pi_i^{A_x} = \frac{\delta \mathcal{L}_{BF}}{\delta(\partial_t A^i_x)} = B_i, \quad \pi_i^{A_t} = \frac{\delta \mathcal{L}_{BF}}{\delta(\partial_t A^t_x)} = 0 \tag{3.18}$$

Where the latter result follows from the equations of motion. The canonical 1-form can then be written as:

$$\Theta = \int_\Sigma q^i \wedge \delta p^i \Rightarrow \Theta = \int_\Sigma \eta_{ij} B^i \delta A^j_x dx = \int_\Sigma \mathrm{Tr}(B \delta A) \tag{3.19}$$

Therefore, the symplectic form $\omega = d\Theta$ is given by

$$\omega = \int_\Sigma \mathrm{Tr}(\delta B \wedge \delta A) \qquad \{A^i_x(x), B^j(y)\} = \delta^{ij} \delta(y - x) \tag{3.20}$$

Which endows the set of all possible configurations of our system with a symplectic structure. This gives us not only the possible physical states, but the entire configuration space of our fields:

$$A \in \Omega^1(\Sigma, SL(2,\mathbb{R})), \quad B \in \Omega^0(\Sigma, SL(2,\mathbb{R})) \tag{3.21}$$

Which span the entire domain of the symplectic manifold, which we denote M_ω. This can also be understood as the collection of tangent bundles on the configuration space, where any pair (A, B) is specified by a point $a \in M_\omega$, and $\pi_a \in T^*_{M_\omega}(a)$. Reducing this space M_ω to the space of states which obey the equations of motion is equivalent to restricting to a submanifold $\mathcal{C}$. Precisely, we define $\mathcal{C}$ as the vanishing locus of the constraints:

$$\mathcal{C} = \{(A, B) \in M_\omega \mid \mathcal{D}_x B = F = 0\} \tag{3.22}$$

Which define possible states of the theory. Recall that the restricting equations on M_ω which define $\mathcal{C}$ are gauge-invariant; that is, if a pair $(A, B) \in T^*\mathcal{A}$ satisfies the constraints

$$F = dA + A \wedge A = 0, \quad \mathcal{D}B = dB + [A, B] = 0, \tag{3.23}$$

then so does the gauge-transformed pair

$$A' := gAg^{-1} + gdg^{-1}, \quad B' := gBg^{-1}, \tag{3.24}$$

for any smooth map $g : \Sigma \to SL(2,\mathbb{R})$, if $F = \mathcal{D}B = 0$ then:

$$F' = dA' + A' \wedge A' = g(dA + A \wedge A)g^{-1} = gFg^{-1} = 0 \tag{3.25}$$

$$\mathcal{D}'B' = d(gBg^{-1}) + [gAg^{-1} + gdg^{-1}, gBg^{-1}] = g\,(dB + [A, B])\,g^{-1} = g\,\mathcal{D}B\,g^{-1} = 0 \tag{3.26}$$

Which is also a point in $\mathcal{C}$. Thus the constraint surface is invariant under gauge transformations. Hence, the orbit of a point $g \in G$ lies entirely within $\mathcal{C}$, and all such points correspond to gauge-equivalent physical configurations. The physical phase space is therefore defined by identifying these gauge orbits:

$$\mathcal{P}_{\text{phys}} = \mathcal{C}/G. \tag{3.27}$$

3.2.1 Symplectic Reduction

As we have shown, points in the physical space of the theory correspond to points in the reduced symplectic space $\mathcal{P}_{\text{phys}}$. For now, let us consider the base space of BF theory as topologically equivalent to a cylinder, $S^1 \times \mathbb{R}$. To explicitly define the physical phase space, we can consider a point in $\mathcal{C}/G$ as an equivalence class of solutions, where we define two solutions as equivalent if they are related as in (3.25), (3.26). Recall the observables of BF theory:

$$[\mathcal{O}_A] = \text{Tr}(\mathcal{P}e^{\oint_\Sigma A^i J_i}) \in G \tag{3.28}$$

$$[\mathcal{O}_B] = \text{Tr}(B^2) \in G \tag{3.29}$$

Which, under some $g(x) : S^1 \to G$, transform as:

$$[\mathcal{O}_{A'}] = \text{Tr}(\mathcal{P}e^{\oint_\Sigma [gA^i g + gdg^{-1}]J_i}) = g(x_0)\text{Tr}(\mathcal{P}e^{\oint_\Sigma A^i J_i})g(x_0)^{-1} \tag{3.30}$$

$$[\mathcal{O}_{B'}] = \text{Tr}(gBg^{-1}gBg^{-1}) = \text{Tr}(B^2) \tag{3.31}$$

Where the first equation follows from a perturbative expansion (For a detailed derivation, see [38]), and the second follows as the trace is cyclic-invariant. Therefore our space of gauge-invariant states is spanned by

$$(\mathcal{O}_{[A]}, \mathcal{O}_{[B]}) \cong G/\mathrm{Ad}(G) \times \mathfrak{g}/\mathrm{Ad}(G) \tag{3.32}$$

Where the second product follows from the fact that $B = gbg^{-1}$ for $b \in \mathfrak{g}$, and it's invariance under $\mathrm{Ad}(G)$ implies any gauge-equivalent elements of $\mathfrak{g}$ are equivalent in the Casimir representation.

Note that since we are now considering equivalence classes under $\mathrm{Ad}(G)$, our gauge-reduced phase space is given by the span of observables $\mathcal{O}_A, \mathcal{O}_B$ under the action of $\mathrm{Ad}(G)$. Specializing now to $G = SL(2, \mathbb{R})$, we have

$$\mathcal{P}_{\text{phys}} = SL(2, \mathbb{R})/\mathrm{Ad}(SL(2, \mathbb{R})) \times \mathfrak{sl}(2, \mathbb{R})/\mathrm{Ad}(SL(2, \mathbb{R})) \tag{3.33}$$

3.3 Residual Boundary Symmetries

The above formula reveals the total space of classical states of SL$(2, \mathbb{R})$ BF theory on the bulk. For the case of $\Sigma \cong S^1 \times \mathbb{R}$, any gauge transformation acting on observables as in (3.24) was shown to leave physical data unchanged (3.25). However, for a general manifold M with nonzero boundary, this is not the full story. In general, a gauge transformation $\mathcal{G} \in \mathrm{Map}(M, G)$ can act nontrivially on the boundary. If the boundary symmetry is larger than the gauge group G, this results in residual symmetries on the boundary which cannot be "gauged away" by local gauge transformations. Formally, local gauge transformations are defined as

$$\mathcal{G}_{\text{local}} := \{g(x) \in \mathrm{Map}(M \to G) \mid g|_{\partial M} = \mathbb{1}_G\}, \tag{3.34}$$

Which form a normal subgroup of the full gauge group $\mathcal{G}$. The remaining symmetry group, $\mathcal{G}_{\text{res}} := \mathcal{G}/\mathcal{G}_{\text{local}}$ hence act nontrivially on the boundary configuration space. These generate dynamical edge degrees of freedom in the classical phase space, and — since they are not homotopic to one another under G_{local} — act nontrivially on physical observables. This is exactly the situation in section 3.2, where the symmetry group of the conformal boundary theory for JT gravity was given by $\text{Diff}(S^1)$. This was then "broken" by the $\text{SL}(2, \mathbb{R})$ isometry group of the bulk, giving rise to the Schwarzian modes on the boundary. In section 4, we will return to quantize these modes.

4 Canonical Quantization of BF Theory

This section provides an outline of the quantization procedure for $\text{SL}(2, \mathbb{R})$ BF Theory, largely adapting the work of [39, 40, 41, 35]. We begin with further analysis on the (3.33), decomposing the classical space into its gauge-invariant sectors. Then, we apply a gauge-fixing procedure to explicitly derive the quantum Hilbert space of the theory on the cylinder, as well as the residual symmetries and algebra which arise for a noncompact manifold.

4.1 BF theory on a Cylinder

For now, we will treat BF theory as defined on $M \cong S^1 \times \mathbb{R}$, where time is noncompact. To begin to dissect the phase space of this theory, note that any state Ψ defined in $\mathcal{P}_{\text{phys}}$

must be conjugation-invariant from the quotient by the adjoint action. That is,

$$\Psi[g] = \Psi[h^{-1}gh] \quad \text{for } \; h \in \mathrm{SL}(2\mathbb{R}) \tag{4.1}$$

Where $[g]$ indicates the gauge-dependence of Ψ. From standard representation theory techniques [42], these gauge-invariant orbits of $\mathrm{SL}(2,\mathbb{R})$ are given by the Cartan subalgebras. This is identical to how the $SU(2)$ symmetry of the electron reduces states $\alpha \in \mathrm{SU}(2)$ to functions of J_z. Unlike for $\mathrm{SU}(2)$, the algebra $\mathrm{SL}(2,\mathbb{R})$ has two orthogonal Cartan subalgebras, isomorphic to $\mathrm{U}(1)$ and $\mathbb{R}^+$, respectively. Geometrically, these subalgebras can be understood as commuting generators of $\mathrm{ISO}(\mathrm{AdS}_2) \cong \mathrm{SL}(2,\mathbb{R})$, where

$$H_1(\nu) = \begin{bmatrix} \cos\nu & \sin\nu \\ -\sin\nu & \cos\nu \end{bmatrix}, \quad H_2(\eta) = \begin{bmatrix} e^{\eta} & 0 \\ 0 & e^{-\eta} \end{bmatrix} \tag{4.2}$$

Correspond to rotations around a timelike direction and boosts along a spacelike direction respectively. Hence points $(\nu, \eta) \in H_1, H_2$ correspond bijectively to the orbits of some $g \in \mathrm{SL}(2,\mathbb{R})$ under conjugation. Therefore one labels states Ψ by their conjugate elements in these Cartan subalgebras:

$$\Psi[g] = \begin{cases} \zeta_1(\nu), & \text{for } [g] \sim \nu \in H_1 \\ \zeta_2(\eta), & \text{if } [g] \sim \eta \in H_2 \end{cases} \tag{4.3}$$

Which has an inner product given by the Haar measure:

$$\langle \zeta_1, \zeta_1' \rangle = \int_0^{2\pi} d\nu \, \sin^2\nu \, \overline{\zeta_1'(\nu)} \, \zeta_1(\nu), \tag{4.4}$$

$$\langle \zeta_2, \zeta_2' \rangle = \int_0^{\infty} d\eta \, \sinh^2\eta \, \overline{\zeta_2'(\eta)} \, \zeta_2(\eta). \tag{4.5}$$

On a state Ψ. Note that the second term of $\mathcal{P}_{\text{phys}}$ corresponding to the Casimir is not needed to define the state, since specifying $[g]$ is enough to specify both $\{\nu, \eta\}$ and project onto ζ_2. Hence any state of the observables (3.30), (3.31) is given by $\Psi[g]$.

This expression includes contributions from both elliptic (ν) and hyperbolic (η) conjugacy classes. In the classical geometry of BF theory, these arise as the generators of the holonomy observable (3.30), where the conjugacy parameter ν corresponds to rotation-like (elliptic) geodesics, and η corresponds to boost-like (hyperbolic) geodesics. These trace out spacelike and timelike paths, respectively, in the internal AdS_2 geometry. For the sake of this paper, we are intersted in $\text{SL}(2,\mathbb{R})$ BF theory as it applies to understanding quantum JT gravity, and hence we will herein restrict all analysis to hyperbolic modes, considering spacelike (ν) states as non-physical. Analytically, this corresponds more precisely to $\text{SL}(2,\mathbb{R})^+$ BF theory, which has been shown to be a reformulation of JT gravity after considering bulk symmetries [43, 44].

As such, a general state in $\mathcal{H}_{\text{phys}}$ is given as a square-integrable function over η:

$$\Psi(\eta) \in L^2(\mathbb{R}^+, \sinh^2 \eta \, d\eta). \tag{4.6}$$

Which gives a valid decomposition of $\mathcal{H}_{\text{phys}}$.

4.1.1 Casimir Decomposition

Recall that $\text{SL}(2,\mathbb{R})$ can be written in terms of basis generators J_i (3.4). Continuing the analogy to $\text{SU}(2)$, recall that states of an $\text{SU}(2)$ gauge theory are labeled by spin-j irreducible representations (irreps) ψ_j for $j \in \mathbb{Z}_{\geq 0}$ and have $\hat{L}^2\psi_j = j(j+1)\psi_j$ as a conserved quantity labeling the states. In an analogous fashion, one may use Kirillov's orbit method to decompose $\mathrm{SL}(2,\mathbb{R})$ into irreps labeled by $s \in \mathbb{R}^+$, where the corresponding conserved quantity is the casimir, C, which is related to s by:

$$\hat{C}\psi_s = \left(s^2 + \frac{1}{4}\right)\psi_s \quad s \in \mathbb{R}^+ \tag{4.7}$$

Which labels the irrep. Just as $\{\theta, \varphi\}$ parameterize irreps of $\mathrm{SU}(2)$, $\{\eta, \nu\}$ parameterize irreps of $\mathrm{SL}(2,\mathbb{R})$, though —since $\mathrm{SL}(2,\mathbb{R})$ is noncompact — irreps do not admit a finite basis as $m \in \{-j, ..., j\}$ does for $\mathrm{SU}(2)$. To represent a given state ψ which has $\mathrm{SU}(2)$ symmetry, one can express an infinite sum over the irreps, as given by the Peter-Weyl Theorem (See Appendix A):

$$\psi \in \mathrm{L}^2(\mathrm{SU}(2)) = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} \mathcal{H}_j \otimes \mathcal{H}_j^* \tag{4.8}$$

Where each $\mathcal{H}_j$ is spanned by its $(2j+1)$ dimensional basis. In the case of $\mathrm{SL}(2,\mathbb{R})$, the analogous approach to derive the form of $\mathcal{H}_{\text{phys}}$ begins with a spectral decomposition of the $\{\eta\}$-basis, expressed in terms of Casimir eigenstates. This is achieved via the Plancherel transform — the analog of the Fourier transform for noncompact Lie groups — which decomposes class functions on conjugacy classes into a continuous sum over irreducible representations:

$$\zeta_2(\eta) = \int_0^\infty ds\, \psi(s)\, \chi_s(\eta), \quad \psi(s) = \int_0^\infty d\eta\, \sinh^2\eta\, \zeta_2(\eta)\, \overline{\chi_s(\eta)}, \tag{4.9}$$

where $\chi_s(\eta)$ is the character of the principal series irrep with Casimir eigenvalue $C = s^2 + \frac{1}{4}$, and $\psi(s)$ forms the spectral data defining a state in the representation $\mathcal{H}_s$. The inner product becomes diagonal in this basis:

$$\langle \zeta_2, \zeta_2' \rangle = \int_0^\infty ds\, \mu(s)\, \overline{\psi(s)}\, \psi'(s), \quad \mu(s) = s \sinh(2\pi s). \tag{4.10}$$

Which defines $\mathcal{H}_{\text{phys}}$ as the space of square-integrable spectral wavefunctions $\psi(s)$, each describing a quantum state with fixed Casimir eigenvalue. The full Hilbert space is thus a continuous superposition of irreducible $\mathrm{SL}(2,\mathbb{R})$ representations — the noncompact analog of the Peter–Weyl theorem — with Plancherel measure $\mu(s)$ governing the density of states. This space encodes all possible quantum geometries of BF theory on the Lorentzian cylinder:[13]

$$\Psi[g] \in \mathcal{H}_{\text{phys}} = \int_0^\infty {}^{\oplus} \mathcal{H}_s\, \mu(s)\, ds, \tag{4.11}$$

$$\text{with} \quad \Psi = \{\psi_s\}_{s \in \mathbb{R}^+}, \quad \text{such that} \int_0^\infty \|\psi_s\|^2_{\mathcal{H}_s}\, \mu(s)\, ds < \infty. \tag{4.12}$$

4.2 Gauge Fixing

Before we begin this section, it will be fruitful to discuss the implications of "gauge fixing". In full generality, one may perform the BRST quantization [45, 46] of a theory to arrive at a quantized Hilbert space. For a rigorous approach to the BRST formalism for topological QFTs, see [47, 48, 49], or [39] for a direct application to $\mathrm{SL}(2,\mathbb{R})$ BF theory.

[13] Strictly speaking, $\mathcal{H}_s^*$ is the contragradient representation defined by $\pi_s^*(g) := \pi_s(g^{-1})^\dagger$. Since BF theory is conjugation-invariant, the decomposition simplifies to $\int \mathcal{H}_s$, as conjugation effectively traces over the dual component — sit is included here only for clarity

The choice of gauge thus has no impact on the observables and invariants of the theory, and hence we will work in a gauge-fixed regime for the sake of pedagogical clarity,[14] however different gauges do produce different spectra of quantum observables. This is a principle which will be used to great effect in the next two sections, as we vary our choice of gauge to uncover various dynamics within the quantized $\mathrm{SL}(2,\mathbb{R})$ BF theory.

4.3 Quantization in the $n \cdot B = 0$ gauge

Having derived the Casimir decomposition of the physical Hilbert space in Section 4.1.1, we now realize that structure via canonical quantization of JT gravity in the $n \cdot B = 0$ gauge. As we have seen in (4.2), the conjugacy orbits of any $g \in \mathrm{SL}(2,\mathbb{R})$ lie in either the elliptic (ν) or hyperbolic (η) orbits. The orthogonality of these orbits can be seen by the fact that $\mathrm{Tr}(g)$ is invariant under conjugation, and hence for any given g, either:

$$g \sim g_v \in H_1(\nu) \Rightarrow \mathrm{Tr}(g) = |\mathrm{Tr}(g_\nu)| < 2 \tag{4.13}$$

$$g \sim g_\eta \in H_2(\eta) \Rightarrow \mathrm{Tr}(g) = |\mathrm{Tr}(g_\eta)| > 2 \tag{4.14}$$

Implies g is in either of the Cartan subalgebras[15].

Hence elliptic elements lie in an $\mathrm{SO}(2)$ subalgebra, and hyperbolic elements lie in an $\mathbb{R}^+$-type subalgebra. Recall that for the sake of application to JT gravity, we are again only interested in $|\mathrm{Tr}(B^2)| > 2$, which will be the object of study in this section. Hence gauge

[14] In the language of the BRST formalism, gauge fixing is understood as an arbitrary transverse "slice" of the gauge orbits.

[15] Note that if $\mathrm{Tr}(g) = 2$, this corresponds to parabolic (lightlike) orbits which are not semisimple. We return to this case at the end of the subsection.

condition $n^a B_a = 0$ (with fixed internal vector n^a) restricts B to lie in a two-dimensional hyperplane of $\mathfrak{sl}(2,\mathbb{R})$, orthogonal to n. For $\pm\mathrm{Tr}(B^2) > 0$, one can choose n to be spacelike as

$$B_{TL} \sim (B_0, 0, 0) \to n \sim (0, n_1, n_2) \Rightarrow B_{TL} = B^0 J_0 \tag{4.15}$$

The physical data thus reduce to a pair (B, A_x) modulo simultaneous adjoint action. Within the Coulomb gauge, the condition $[A_x, B] = 0$ implies that A_x must lie in the stabilizer subalgebra of B, which is 1-dimensional for non-lightlike B, corresponding to the subalgebras generated by (η) for timelike trajectories, respectively.[16] This simplifies the holonomy to the expression:

$$\mathrm{Hol}_\gamma = e^{\oint_\gamma dx T(x) J_0} \tag{4.16}$$

Where $T(x)$ is the gauge-invariant parameter, and J_0 is the generator of boosts. Explicitly, these are given by:

$$T(x)_{TL} := Q = \frac{B^a A_a}{B^a B_a} \xrightarrow{n\cdot B = 0} \frac{B^0 A_0}{B^0 B_0}, \quad \tau_* = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.17}$$

Therefore (4.16) reduces to

$$H_I = e^{-J_0 \eta}, \quad \eta = \oint_{S^1} B^0 dx \tag{4.18}$$

Such that we may define a conjugate momenta operator $\pi_\eta := \frac{1}{2} B^a B_a$, where $\{\hat{\pi}_\eta, \eta\} = 1$ follows from the equations of motion. Thus gives the canonical quantization

$$[\hat{\pi}_\eta, \eta] = i\hbar, \; \hat{\pi}_\eta = i\hbar \frac{d}{d\eta} \tag{4.19}$$

[16]Note no path-ordering is needed since each sector is abelian.

With eigenfunctions $\varphi_\omega = e^{i\omega\eta}$ (i.e.plane waves). As in the case of a free particle in standard quantum mechanics, one might expect wavefunctions to be elements of $L^2(\mathbb{R})$, i.e., square-integrable functions $\psi(\eta)$ with continuous eigenvalues $\omega \in \mathbb{R}$. However, there is a subtle difference. Unlike in quantum mechanics, BF theory has observables which are entirely topological, and as such only depends on ω through its exponentiated form. This corresponds to the Pontryagin dual group of $\mathbb{R}$, comprised of elements $\omega \in \mathrm{Hom}(\mathbb{R}, S^1) := \mathbb{R}_B$, which is topologically distinct from $\mathbb{R}$ since each element defines a unique "dimension". It is referred to in the literature as the Bohr compactification $\mathbb{R}_B$ of $\mathbb{R}$. The characters of $\mathbb{R}_B$ are "almost-periodic" functions on S^1, and have a character decomposition

$$\psi_s(\omega) = e^{-is\omega/\hbar} \quad \text{for } \omega \in \mathbb{R}_B,\ s \in \mathbb{R} \tag{4.20}$$

$$\int_{\omega\in\mathbb{R}_B} d\mu(\omega)\psi_s(\omega) = \delta_{\omega 0} \tag{4.21}$$

Where $\psi_s(\omega)$ is the Fourier dual of φ_ω, and the measure on $\mathbb{R}_B$ is derived in the Appendix. This space is non-separable, reflecting the topological character of the degrees of freedom, and supports a unitary representation of the Heisenberg algebra with dense frequency spectrum.

Quantizing the orbit for fixed Casimir $C = s^2 + \frac{1}{4} := \omega^2 + \frac{1}{4}\hbar^2$ in this framework produces a Hilbert space $\mathcal{H}_s$ given by

$$\varphi_s(\eta) = \int_{\mathbb{R}_B} a(\omega) e^{i\omega\eta/\hbar} d\mu(\omega) \tag{4.22}$$

$$\tilde{\Psi}(\omega) = \sum_s \tilde{a}_s e^{-is\omega/\hbar} \tag{4.23}$$

in the η, ω basis respectively.

4.4 Quantizing the Schwarzian

Now that we have seen the quantization of the bulk states of $\mathrm{SL}(2,\mathbb{R})^+$ BF theory, it will be fruitful to consider a different gauging procedure from which we may extract the dual boundary theory. As a topological theory, $\mathrm{SL}(2,\mathbb{R})^+$ BF theory has no dynamics in the bulk, hence we wish to understand the boundary correspondence to these modes, which we will see reproduces the same Schwarzian action (2.34) as derived in the semiclassical JT picture. Up to now, we have considered BF theory as a purely-topological bulk theory with no boundary. In the case where one has a nonzero boundary with S^1 topology, the full BF theory expression is given by:

$$S_{BF} = \int_M \mathrm{Tr}(BF) + \frac{1}{2} \int_{\partial M} \mathrm{Tr}(BA_\theta) d\theta \tag{4.24}$$

Which will be necessary to consider topologies useful in JT gravity. To consider BF theory as a reformulation of JT gravity, we herein will treat $\mathrm{SL}(2,\mathbb{R})^+$ BF theory on a Euclidean manifold, where the Wick rotation $t \to i\tau$ and coordinate change $\rho \to 1/z$

transform the Robinson-Bertotti metric (2.13) as:

$$ds^2 = Q^2 G\left(\frac{dz^2 + d\tau^2}{z^2}\right) \tag{4.25}$$

Which is the Poincaré patch of AdS_2 on which we will consider BF theory. Topologically, this is equivalent to capping off the conformal boundary at $z \to 0$, which in Euclidean coordinates has compactified time β. At $z \to \infty$, one approaches the horizon of the black hole, which in the Euclidean signature corresponds to "pinching" off the geometry. This topology is colloquially referred to as the "cigar" geometry:

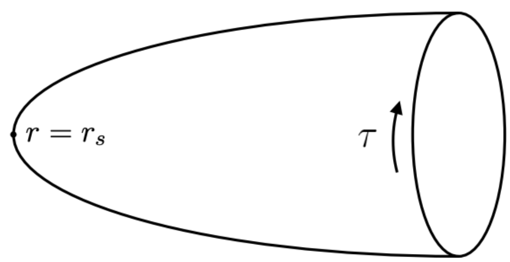

Figure 3: An illustration of the cigar geometry, with compact Euclidean time τ and a smooth cap at the horizon (Photo from [50])

When we define BF theory on a manifold with boundary, the metric restricted on the boundary cannot fully be "gauged away" as in the bulk. Analogous to the $\mathrm{Diff}^+(S^1)/\mathrm{SL}(2,\mathbb{R})$ residual freedom seen in JT gravity, the general residual symmetry group which remains in BF theory (as given by (3.3)) provides:

$$\mathcal{G}_{\mathrm{res}} \cong \mathcal{G}_{\mathrm{bdry}}/\mathcal{G}_{\mathrm{Bulk}} \tag{4.26}$$

As the residual boundary symmetries. A standard fact from Complex analysis is that this Conformal disc has an isometry class given by $\mathrm{PSL}(2,\mathbb{R})$. Astonishingly, taking the total space of boundary parameterizations $f(\tau) \in \text{Diff}(S^1)$ up to equivalence under bulk isometry immediately yields the same Schwarzian modes as seen in (2.34):

$$\text{Diff}(S^1)/\text{PSL}(2, \mathbb{R}) \cong |f\rangle_{\text{Sch}} \tag{4.27}$$

As we will see later chapters, this framework will prove to be considerably powerful when considering manifolds with much more complicated internal bulk structure. For now, let us consider the quantization of these modes such that we may understand them as dual to the hyperbolic modes as seen in (4.12), (4.21).

4.4.1 BF THEORY ON THE BOUNDARY: THE SPATIAL GAUGE

In order to quantize these modes, it will be useful to perform a gauge fixing as before, but now on the conformal disc. From the general form of BF theory, recall the discussion of the classical phase space of BF theory, the bulk condition $F = 0$ implies that the connection is a "pure gauge": $A = gdg^{-1}$, and the phase space is given by the space of flat connections modulo equivalence under the $\text{SL}(2, \mathbb{R})$ bulk gauge. Considering the disc $\mathbb{D}$ as parameterized by the coordinates (r, θ), we work in the gauge where $A_r = 0$. This gives

$$A = A^i_\theta J_i d\theta \tag{4.28}$$

As the pure connection along the boundary, which encodes the full information of the theory. Analogous to the classical BF theory decomposition, the holonomy around the conformal boundary is given as:

$$\mathrm{Hol}_{\mathrm{bdry}} = \mathcal{P}\exp(\oint_{S^1} A_\theta d\theta) \tag{4.29}$$

Where $A_\theta = g(\theta)dg^{-1}(\theta)$ parametrizes A. This gives a space of flat configurations indexed by $g(x) : S^1 \to \mathrm{SL}(2,\mathbb{R})$, such that $g \sim g'$ if they are related by conjugation by an element $h \in \mathrm{SL}(2,\mathbb{R})$. This follows directly from (3.30), where

$$\mathcal{O}_H(g') = h^{-1}\mathcal{O}_H h \Leftrightarrow g(x) = h^{-1}g'(x)h \tag{4.30}$$

So the full space of gauge-inequivalent flat connections is given by

$$\mathcal{M}_{\mathrm{flat}} \cong \mathrm{Map}(S^1, \mathrm{SL}(2,\mathbb{R}))/\mathrm{SL}(2,\mathbb{R}) \cong \mathrm{Diff}(S^1)/\mathrm{PSL}(2,\mathbb{R}) \cong \{|f\rangle_{\mathrm{Sch}}\} \tag{4.31}$$

Which is a deep correspondence arising from Teichmüller geometry (See: [51, 52]).[17] This is the manifestation of the pure AdS/CFT correspondence realized in $\mathrm{SL}(2,\mathbb{R})$ BF theory. Each bulk gauge equivalence class is dual to a Schwarzian state on the boundary.

4.4.2 The Virasoro Algebra in $\mathrm{SL}(2,\mathbb{R})^+$ BF theory

The correspondence can be understood more clearly once the Schwarzian partition function is known explicitly. Recall from the JT gravity discussion that the Schwarzian action was given by

$$S[f] = -C\int_0^\beta dt\,\{f(t), t\} \tag{4.32}$$

[17] Herein we refer to $\{|f\rangle\}_{\mathrm{Sch}} := \mathcal{S}$ in line with the Teichmüller literature.

Which has a natural symplectic form, given by $S(u) \sim \{f(u), u\}$ with conjugate generator $\delta f(u)$:

$$\Omega = \int_{S^1} \delta S(u) \wedge \delta f(u) \tag{4.33}$$

In essence, one can consider a Schwarzian mode $T_f(u) \propto \{f(u), u\}$ as a Laurent series over a $\mathbb{Z}$-indexed basis modes:

$$T_f(u) = \sum_n L_n e^{inu}, \quad L_n = \frac{1}{2\pi} \int_{S^1} du e^{-inu} T_f(u) \tag{4.34}$$

$$\{L_m, L_n\} = (m-n) L_{m+n} \tag{4.35}$$

This is precisely equivalent to the Virasoro algebra with zero central charge (See Appendix C). Hence, the formalism of such modes applies to give a Hilbert space on quantized modes of the Schwarzian spectrum. As it turns out, these Virasoro modes have a natural decomposition by the quadratic Casimir of the Virasoro algebra, given by:

$$\mathcal{C} = L_0^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) \tag{4.36}$$

For $\mathrm{SL}(2, \mathbb{R})$. This follows from $\mathcal{C}$ being the universal enveloping algebra which commutes with all generators. This diagonalizes the representations of $\mathrm{SL}(2, \mathbb{R})$ into the principal series

$$\mathcal{C} \left| f \right\rangle = \left(s^2 + \frac{1}{4} \right) \left| f \right\rangle, \quad s \in \mathbb{R}^+ \tag{4.37}$$

Which is precisely the decomposition given by (4.12). It is no accident, then, that

$$|f\rangle \in \mathcal{H}_{\text{Sch}} = \int_0^\infty ds \mu(s) \mathcal{H}_s, \quad s \in \mathbb{R}^+ \tag{4.38}$$

$$\text{with} \quad \Psi = \{\psi_s\}_{s \in \mathbb{R}^+}, \quad \text{such that} \int_0^\infty \|\psi_s\|^2_{\mathcal{H}_s} \mu(s)\, ds < \infty. \tag{4.39}$$

Where $\mu(s) = s \sinh(2\pi s)$ is the same Plancherel measure in (4.12). This should not be a surprise: The Schwarzian states under the correspondence (4.31) are hence dual to the same principal series representation of $\mathrm{SL}(2,\mathbb{R})^+$, and thus should admit the same Casimir decomposition.

Furthermore, restricting the space of Schwarzian modes to those with fixed energy $\{|f\rangle_s\}$ has the same effect as the Coulomb-gauge fixing procedure had on the fixed-Casimir states Ψ_s, which under (4.21) decomposed into a finite subset of $\mathbb{R}_B$-indexed modes $\{\psi_s\}$. Explicitly, under the noncompact Peter-Weyl theorem, one has that any continuous spectra indexed by its characters on the maximal torus $\mathbb{T}^n$, where the characters are determined by the representation group. In the case of the principal series representations of $\mathrm{SL}(2,\mathbb{R})$, we saw in (4.21) that these characters were given by $\{e^{i\omega\eta/\hbar}\}$ for $\omega \in \mathbb{R}_B$. The Schwarzian modes hence have an equivalent representation:

$$|f_s\rangle = \int_{\mathbb{R}_B} \tilde{a}_s(\omega)\, e^{i\omega\eta/\hbar}\, d\mu(\omega) \tag{4.40}$$

Where η is conjugate to the Casimir, just as in the case of (4.21). We thus have a bulk-boundary duality at three levels: The full quantum Hilbert space, generic states, and fixed-casimir states:

$$\mathcal{H}_s^{\mathrm{BF}} \;\cong\; \mathcal{H}_s^{\mathrm{Sch}} \;\cong\; L^2(\mathbb{R}_B, d\mu(\omega)) \tag{4.41}$$

$$\Psi = \{\psi_s\} \Leftrightarrow |f\rangle = \{|f\rangle_s\} \tag{4.42}$$

$$\Psi_s(\eta) = \int_{\mathbb{R}_B} a_s(\omega)\, e^{i\omega\eta/\hbar} d\mu(\omega) \;\;\Leftrightarrow\;\; |f_s\rangle = \int_{\mathbb{R}_B} \tilde{a}_s(\omega)\, e^{i\omega\eta/\hbar}\, d\mu(\omega) \tag{4.43}$$

Where the Plancherel measure $\mu(s) = s\sinh(2\pi s)$ governs the spectral decomposition of both theories:

$$\mathcal{H}_{\mathrm{phys}}^{\mathrm{BF}} = \mathcal{H}_{\mathrm{Sch}} = \int_0^\infty \mathcal{H}_s\, \mu(s)\, ds. \tag{4.44}$$

We retain duality. Thus, the Bohr–Fourier decomposition of the bulk holonomy modes and the quantized boundary Schwarzian modes are two descriptions of the same topological spectrum, unified by principal series decomposition — and corresponding Pontryagin dual — on on $\mathrm{SL}(2,\mathbb{R})^+$.

5 Nonfactorization of Wormholes in JT gravity

Up to now, we have considered $\mathrm{SL}(2,\mathbb{R})^+$ BF theory as classically equivalent to JT gravity as a model for the asymptotic regime of Reissner-Nordström Black holes. From this vantage point, we have seen that BF theory reproduces the quantized dynamical degrees of freedom in the bulk as dual to the Schwarzian modes which arise as residual symmetries on the boundary. In this sense, one has a clear, unique "holographic dual" theory to bulk JT dilaton gravity as the residual Schwarzian modes, with a one-to-one correspondence in the respective partition functions. However, there is much more to the story of

JT gravity. In the following chapters, we will explore the full range of possible states of this theory, uniting Black hole thermodynamics and Entanglement entropy in the process. In this more general picture, the question of a unique holographic dual is muddled by the divergence of JT gravity's dynamics the traditional AdS/CFT picture: As we shall see, the dual theory is much more subtle than one might expect from the Cigar topology. We begin with a brief overview of the motivation for considering more complex topologies, motivated by the path integral formulation of quantum mechanics. From here, we will see how a weighted sum over topologies arises naturally in the study of quantum gravity, acknowledging the subtleties that arise along the way.

5.1 The Euclidean Gravitational Path Integral

In quantum mechanics, recall that the Feynman path integral [53] computes the transition amplitude between two spacetime points as a sum over all continuous paths:

$$\langle x_f, t_f | x_i, t_i \rangle = \int \mathcal{D}[x(t)]\, e^{iS[x(t)]} \tag{5.1}$$

Each path $x(t)$ is a distinct line in $\mathbb{R}^{3+1}$ from (x_i, t_i) to (x_f, t_f). All paths are homotopic to one another, yet contribute distinctly to the path integral. In the Wick-rotated theory, the famous "constructive interference" around $\delta S = 0$ corresponds to the minimum of the exponent $e^{-S_E[x(t)]}$, where off-shell paths are exponentially suppressed by their relative action. That is to say, if γ_0 is the classical trajectory with action S_0, and γ_1 is an arbitrary path with action $S_1 = S_0 + \delta S$, the Euclidean path integral reads:

$$\frac{\mathrm{P}(\gamma)}{\mathrm{P}(\gamma_0)} = \frac{e^{-S_1}}{e^{-S_0}} = e^{-\delta S} \tag{5.2}$$

The central question of quantum gravity is the supposed existence of a "gravitational path integral". This hypothetical object would then determine the probability of a given set of boundary conditions from an integral over the bulk. The natural conjectured object is then:

$$\langle g_{\mu\nu,\text{final}} | g_{\mu\nu,\text{initial}} \rangle \stackrel{?}{=} \sum_{\text{compatible geometries } \mathcal{M}} e^{\frac{i}{\hbar}S} \text{ with } S = \int_{\mathcal{M}} d^4x \sqrt{-g}\mathcal{R} \tag{5.3}$$

Where "compatible geometries" $\mathcal{M}$ hence counts all bulk configurations which restrict to $g_{\mu\nu,\text{final/initial}}$ on the boundary.[18] From the path integral perspective, the inclusion of interior bulks with inequivalent topologies is a natural continuation of the logic, somewhat akin to the perturbative n-expansion in Quantum Field Theory.

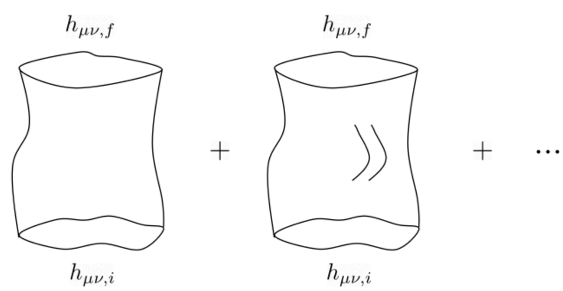

Figure 4: nontrivial topologies, though suppressed, are still counted in $\mathcal{Z}_{\text{grav}}$. (Photo from [19])

Under a Wick rotation, minimal action amplitudes in quantum mechanics become "saddles" over the configuration space, and are the paths which survive the semiclassical

[18]The credit for this succinct expression (5.3) of the conjectured quantum-gravitational path integral is due to [19].

limit. Similarly, in a Wick-rotated theory of quantum gravity, one expects that the amplitudes which determine semiclassical behavior are the minimal-action configurations of the bulk.[19] More formally, the above conjecture leads one to the Euclidean path integral in quantum gravity, often expressed as a partition function $\mathcal{Z}_{\text{grav}}$. For an n-dimensional manifold Σ, it is defined by boundary conditions $\mathcal{B}$ on $\partial\Sigma$; the interior is integrated as:

$$Z[\mathcal{B}] = \sum_i \int_{\mathscr{M}(\Sigma^i)} \int \frac{\mathcal{D}g\, \mathcal{D}\Psi^i}{\text{Diff}(\Sigma^i)} e^{iS_{\text{total}}[g,\Psi^i]} d\mu \tag{5.4}$$

Where $\mathscr{M}(\Sigma)$ is the moduli space of Σ^i and i indexes the possible topologies with boundary $\mathcal{B}$ — this counts all possible geometries on which a theory with wavefunctionals Ψ^i can be defined. Note that one also quotients this integral by $\text{Diff}(\Sigma^i)$ as to not count states related by a Lorentz transform.

In the case of JT gravity (defined on a 2-manifold Σ). The path integral becomes a sum over 2-manifolds with g handles and n boundaries such that the conformal boundaries are all of identical length. The formula obtained from the Euclidean path integral for this is given by:

$$\mathcal{Z}_{\text{grav}} = \sum_{g=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} Z_{g,n}(\beta_1, ..., \beta_n) \tag{5.5}$$

$$Z_{g,n}(\beta_1, ..., \beta_n) = \int_{\mathscr{M}_{g,n}} \mathcal{D}g\mathcal{D}\phi e^{-S_{JT}[g,\phi]} \tag{5.6}$$

Where $\{\beta_i\}$ are understood to be the thermal conformal boundaries, which are taken to infinity in the semiclassical limit. $\chi_{g,n}$ is the Euler characteristic of the given topology,

[19]Note that this is why higher-genus topologies are not seen in the semiclassical limit, as they are suppressed by an action which depends on the Euler characteristic of the topology.

given by the Gauss-Bonnet theorem as $\chi_{g,n} = 2g + n - 2$ which reflects the topological term in JT gravity. Generically, the integration is performed over all possible bulk geometry configurations, which is given by the moduli space $\mathscr{M}_{g,n}$. As we will see in later sections, off-shell configurations not satisfying the $\mathcal{R} - 2 = 0$ constraint of JT gravity are exponentially suppressed, and the integration measure will reduce to a measure over globally hyperbolic manifolds.[20]

This expression raises critical questions in JT gravity. For an arbitrary connected topology, since the boundaries $\{\beta_i\}$ are connected, the possible Schwarzian states do not correspond bijectively to the bulk states as they did for the Euclidean disc. This will become apparent in the following section, but it conceptually follows from the fact that the on-shell configurations of JT gravity live on globally hyperbolic bordered manifolds, which are not well-defined for boundary metric given by an arbitrary Schwarzian mode $|f\rangle_i$. For a single-boundary disc, this is not a problem: One can always perform a global gauge transformation to ensure the bulk is well-defined and globally hyperbolic. However, for a multi-boundary system, the connectedness of the bulk complicates this procedure. We will see in the following sections the interpretations which arise from this complication, as well as some proposed resolutions of this issue.

A second issue with this so-called "genus expansion" formula for quantum gravity is the divergence at large g. In quantum field theory, the number of interaction diagrams

[20]This measure is known as the Weil-Petersson volume of a generic manifold, and will be discussed in more detail in section 6.

grows factorially at large n, while the scattering amplitude from any given diagram decays exponentially in n, leading to a divergence. This birthed the study of renormalization — a methodology for treating these divergences systematically [54, 55, 56]. In Euclidean quantum gravity, attempts to make sense of the same divergence have come in a wide variety of approaches. Original works on this divergence attempted to generalize the Renormalization Group techniques to describe gravity [57], while other ideas (such as Loop quantum gravity and SUSY string theory) have done away entirely with the issue, treating gravity as a separate UV-complete theory [58, 59].[21].

Recent interest, however, has turned in another direction: Understanding JT gravity as dual to Random Matrix Theory, where UV-complete behavior is understood nonperturbatively as the behavior of a double-scaled random matrix integral. The seminal work by Saad, Shenker, and Stanford (SSS) [1] demonstrate an exact dual to JT quantum gravity in the form of a double-scaled random matrix integral. Their work has reignited interest in statistical models low-dimensional models of quantum gravity, especially the Sachdev-Ye-Kitadev (SYK) model [60] which was shown by [1] to be the unique dual of JT gravity. The details of this model are included in the appendix for reader convenience; for a thorough review, see [61, 62, 22].

As a consequence of this striking duality proposed by Saad, Shenker, and Stanford (SSS), many new questions have arisen regarding the nature of AdS/CFT and its propo-

[21]The issue of EQC renormalization is beyond this paper; the interested reader is referred to the citations for a more thorough understanding of this issue.

sition of a unique holographic dual. While traditional AdS/CFT duality suggests that a bulk theory should have a unique, factorizing holographic dual, the derivation of SSS suggests an alternative hypothesis: The random matrix formulation of JT gravity suggests that the dual theory is not unique, rather, it is an "ensemble" of quantum-mechanical Hamiltonains, suggesting an "ensemble-averaged" holographic dual is the natural interpretation of Euclidean JT quantum gravity.

Formally, their model suggests that $\mathcal{Z}_{\text{grav}}$ on an arbitrary manifold[22] factorizes as a product of "trumpet" geometries, each glued onto a bulk manifold such that the system generates the saddle points in the Euclidean gravitational path integral:

$$Z_{g,n}(\beta_1, ..., \beta_n) = \int \left[\prod_{i=1}^{n} b_i db_i Z_{\prec}(\beta_i, b_i)\right] V_{g,n}(b_1, ..., b_n) \tag{5.7}$$

$$Z_{\prec}(b_i, \beta_i) = \sqrt{\frac{\gamma}{2\pi\beta_i}} e^{-\gamma b_i^2/2\beta_i} \tag{5.8}$$

Where $Z_{\prec}(\beta_i, b_i)$ is the "trumpet" geometry with a thermal boundary of length β_i and minimal geodesic length b_i, and $V_{g,n}$ is the Weil-Petersson volume which is the measure of the moduli of hyperbolic manifolds which satisfy the $\mathcal{R} = -2$ equation of motion in JT gravity[23]. This is the ensemble interpretation, where each trumpet is considered as a factorizable "building block" of the partition function.

There is much more to be said about this model, and most of it requires an under-

[22]In the context of this paper, a manifold is understood to be two dimensional with an arbitrary number of boundaries and handles, denoted as (n) and (g) respectively.

[23]More will be discussed on these Weil-Petersson volumes in the following sections. A precise mathematical definition is given in Appendix B

standing of how JT gravity behaves on more complex manifolds. The subtleties that arise in this genus expansion have provoked an ontological split in the meaning of this duality — some authors such as Harlow and Jafferis suggest that the genus expansion is not meaningful, and that the nonfactorization that arises in JT gravity is expected: The connectedness of a multi-boundary topology prevents the naive factorization of the Hilbert space à la AdS/CFT.

The remainder of this section serves to motivate and understand the factorization puzzle in JT gravity. As motivated by the Euclidean gravitational path integral, we begin by introducing JT gravity Euclidean wormholes as natural objects of study, followed by a brief discussion on the novel dynamics that arise when considering quantum-mechanical duals multi-boundary system, suggesting the naive AdS/CFT dual to Euclidean JT gravity as a maximally-entangled thermofield double (TFD) state on the Euclidean wormhole. The factorization problem will then arise naturally as a conequence of JT gravity's entanglement structure as implied by the equations of motion. We examine the manifest nonfactorization from the lens of $\mathrm{SL}(2,\mathbb{R})^+$ BF theory, generalizing the techniques in section 4 to account for the wormhole topology. Finally, we provide a discussion on what this nonfactorization implies for AdS/CFT which will serve as a primer for the ontological split among contemporary interpretations of this puzzle.

5.2 Wormholes as Euclidean Saddle Points

Recall that in the generic formulation of the JT gravitational path integral (5.6), one has a sum over topologically distinct bulk geometries, and for each topology one considers all possible moduli (geometries). In the case of the wormhole (2 boundaries, 0 handles) the contribution is given by: (5.6)

$$\int_{\mathscr{M}_{0,2}} \mathcal{D}g\mathcal{D}\phi e^{-S_{JT}[g,\phi]} \tag{5.9}$$

Where $\mathscr{M}_{0,2}$ is the moduli space of all possible wormhole geometries.[24] Analogous to the case in quantum mechanics where one considers all possible trajectories of a particle, the gravitational path integral takes all possible moduli, and weights then by the exponential of their action, encoded in the partition function. Recall that in quantum mechanics, the principle of stationary action implies that one recovers the classical trajectory as $\hbar \to 0$. This notion applies equally well to the gravitational analog, where one hopes to recover the states of minimal bulk action in the semiclassical limit. To illustrate this, consider a configuration (g_0, ϕ_0) which is locally minimized in the above expression. Perturbing around a minima then yields:

$$S[g,\phi] \approx S[g_0,\phi_0] + \frac{1}{2}\delta^2 S[g_0,\phi_0] + \cdots \tag{5.10}$$

[24]Note that in the specific case of the wormhole, the moduli space is quite small, and is a one-parameter family parameterized by the minimal "throat" of the wormhole.

Such that we recover the condition

$$\left.\frac{\delta S}{\delta g}\right|_{g_0} = 0, \quad \left.\frac{\delta S}{\delta \phi}\right|_{\phi_0} = 0 \tag{5.11}$$

Which define the saddle points of the wormhole contribution. These correspond functionally to solutions of JT gravity which satisfy the equations of motion, hence we are interested in those wormholes which have constant negative curvature, and allow for a dilaton solution $\nabla^2\varphi = \varphi$.[25] These moduli then give the domain of on-shell wormhole solutions, and are precisely those studied by Mirzakhani [63], who showed that the measure over these moduli is precisely given by the Weil-Petersson volumes $V_{g,n}(\beta_1, ..., \beta_n)$, which can be understood as measuring the density of on-shell geometries for a hyperbolic bulk geometry with boundary lengths $\{\beta_i\}$.[26] To evaluate $Z_{JT}(\beta_1, \beta_2)$ for a given on-shell geometry, let us now turn to a discussion on the states which live on such a wormhole.[27]

5.3 JT Gravity Hartle-Hawking States

To understand the structure of states prepared by Euclidean topologies in JT gravity, it is instructive to begin with the simpler case of the Euclidean disk. This geometry corresponds to the Wick rotation of Lorentzian AdS_2, and provides a canonical example of Hartle–Hawking state preparation. On the disk, the JT gravitational path integral reduces

[25]Dilaton profiles have a one-parameter family of solutions over a given point in moduli space, and are absorbed into a constant in the literature

[26]Mirzakhani's work on the Weil-Petersson volume is also mentioned in Appendix B.

[27]For now on, a "wormhole" state of JT gravity will implicitly mean a wormhole with on-shell bulk geometry.

to an integral over Schwarzian boundary modes, with a Plancherel spectral density $\rho(E)$ as derived in (4.44).[28] The resulting partition function is given by:

$$Z_{\text{disc}}(\beta) = \int_0^\infty dE\, \rho(E)\, e^{-\beta E}, \quad \rho(E) = \sinh(2\pi\sqrt{E}) \tag{5.12}$$

where E labels the Casimir eigenvalue of the $\mathrm{SL}(2,\mathbb{R})$ representation, corresponding to the Schwarzian energy. To interpret this partition function as arising from a quantum state, consider the $t = 0$ spacelike slice of Lorentzian AdS_2 obtained by Wick rotating the disk. This slice corresponds to a radial geodesic extending from the center of the disk to the boundary. Along this slice, the Euclidean path integral over the lower half-disk prepares a wavefunctional: this is the Hartle–Hawking (HH) state, which reproduces the thermal partition function as its norm. We now define an orthonormal basis of energy eigenstates $|\Psi_E\rangle$ satisfying:

$$\langle \Psi_E | \Psi_{E'} \rangle = \rho(E)\delta(E - E'). \tag{5.13}$$

Then the Hartle–Hawking state is defined such that its inner product yields the original partition function:

$$|\mathrm{HH}\rangle = \int_0^\infty dE\, e^{-\beta E/2}\, |\Psi_E\rangle\,, \quad Z(\beta) = \langle \mathrm{HH}|\mathrm{HH}\rangle\,. \tag{5.14}$$

This expression exhibits the Hartle-Hawking state as a superposition over energy eigenstates, weighted by a Euclidean evolution factor $e^{-\beta E/2}$. It reflects the standard

[28]The Plancherel measure given in (4.44) is in terms of the holonomy s, where $\mu(s) = s\sinh(2\pi s)$. Performing a Jacobian transformation, and normalizing the Casimir yields the described $\rho(E)$, in line with the literature.

gravitational intuition: the path integral over a smooth Euclidean disk prepares a pure state on a spatial slice, whose Lorentzian continuation describes the vacuum of the AdS_2 black hole geometry. In the case of the wormhole, we perform a similar operation to describe a state along a spacelike slice. In the case of the wormhole, one has two conformal boundaries, with lengths β_L, β_R respectively.

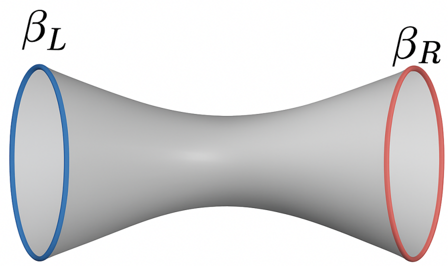

Figure 5: An illustration of a Euclidean spacetime wormhole, where $\beta_{L/R}$ label conformal thermal boundaries in asymptotic AdS space.

The total Hilbert space is generically $\mathcal{H}_L \otimes \mathcal{H}_R$. Unlike in the case of the disc, however, not all states exist as possible states on the JT gravity wormhole.

5.3.1 Using $SL(2,\mathbb{R})^+$ BF theory to probe wormhole states

To determine the possible JT gravity states on the wormhole, we now employ the BF theory framework to this setup. Recall that in BF theory, the observables are given by the holonomy and the Casimir, where the former determines the latter. To study the wormhole in BF theory, we consider the manifold $\Sigma \cong S^1 \times I$, where $I \cong [0,1]$ represents the axial direction of the wormhole, with boundary lengths β_L, β_R respectively. The action is given by (4.24), where the residual symmetries were given as $\mathcal{G}_{\text{res}} \cong \mathcal{G}_{\text{boundary}}/\mathcal{G}_{\text{bulk}}$ in (4.26). In the case of the wormhole geometry, the residual symmetry group is different from the disc: Not only do we have two boundaries which each have reparameterization symmetry, the wormhole breaks the $\text{SL}(2,\mathbb{R})$ symmetry of the bulk down to $\text{U}(1)$. Furthermore, since the boundaries are connected, this symmetry acts diagonally on both the left and right reparameterizations. Hence the residual symmetry group after a choice of gauge is given by:

$$\mathcal{G}_{\text{res}} \cong \frac{\text{Diff}^+(S^1) \times \text{Diff}^+(S^1)}{\text{U}(1)_{\text{diag}}} \tag{5.15}$$

Which, from the analysis of BF theory on the Euclidean disc, would naively suggest that the bulk Hilbert space is dual to this residual symmetry group; that is,

$$\mathcal{H}_{\text{Wormhole}} \overset{?}{\cong} \{|f\rangle_L \otimes |f\rangle_R\} \tag{5.16}$$

Where the Schwarzian modes are now broken by a smaller bulk isometry. However, there is an important difference in $\mathcal{G}_{\text{res}}$ on the wormhole: The $\text{Diff}^+(S^1)$ reparameterizations do not act independently. To see this explicitly in BF theroy, consider the action (4.24) on $\Sigma \cong S^1 \times \text{I}$, where time runs in the compact direction:

$$S_{BF} = \int_{\Sigma} \text{Tr}(BF) + \int_{\partial\Sigma} dt\, \text{Tr}(BA_t) \tag{5.17}$$

Recall that this action is topological in the bulk. Hence, applying a perturbation to the action only gives a nontrivial contribution from the boundary such that

$$\delta S_{BF} = \text{bulk} = 0 + \int_{\partial\Sigma} dt\, \text{Tr}(B\delta A_t - A_t \delta B) \tag{5.18}$$

Following the construction of [44], we use boundary conditions which incorporate the U(1) bulk symmetry — this amounts to setting the generators of U(1) symmetry to zero, such that the generalized boundary conditions are:

$$A_t^{\mathfrak{j}}\Big|_{\partial\Sigma} = B^{\mathfrak{j}}\Big|_{\partial\Sigma} \quad \text{and} \quad A_t^{\mathfrak{h}}\Big|_{\partial\Sigma} = B^{\mathfrak{h}}\Big|_{\partial\Sigma} = 0 \tag{5.19}$$

Where $\mathfrak{j}$ are the U(1) generators, and $\mathfrak{h}$ are the residual symmetries not contained in the bulk. For the disc, this would have been $\mathfrak{j} = \mathfrak{h} \sim \mathfrak{sl}(2,\mathbb{R})$, however now we have $\mathfrak{j} \sim \mathrm{U}(1)$, $\mathfrak{j} \subset \mathfrak{h} \sim \mathfrak{sl}(2,\mathbb{R})$.[29] With these boundary conditions in place, the variation above reduces to

$$\delta S_{BF}[g] = \frac{1}{2}\int_{\partial\Sigma} dt\, \mathrm{Tr}(g^{-1}dt)^2, \quad g(\tau) \in \mathrm{Map}(\mathbb{H}_2^+, S^1) \tag{5.20}$$

Where $\mathbb{H}_2^+ \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$. Note that $\partial\Sigma \cong S^1_{\beta_L} \sqcup S^1_{\beta_R}$ for the wormhole, where the boundary conditions act identically on both. Furthermore, since the holonomy given by some $g(\tau)$ is a topological observable, it must be invariant under such residual symmetries, which are definitionally the quotient class of the global symmetries by the bulk symmetries — this is the critical fact that prevents tensor-product factorization! We now have all of the ingredients to compute the partition function: States are defined as U(1)-broken boundary reparameterizations.

[29]We use $\sim$ to denote that these elements $\mathfrak{g}$ are the generators of the corresponding lie algebra.

5.3.2 Wormhole Thermofield Double States

Let $|\Psi\rangle$ be a state defined on an S^1 boundary of the wormhole. From the reasoning above, if one requires the JT/BF theory equations of motion to be satisfied, it follows that:

$$\hat{C}\,|\Psi_L\rangle = \hat{C}\,|\Psi_R\rangle \Rightarrow |\Psi_L\rangle \otimes |\Psi_R\rangle \sim \rho(E)\delta(E_L - E_R) \tag{5.21}$$

Where $\hat{C}$ is the Casimir operator defined in equation (4.7). Hence, the JT gravity wormhole states are spanned by the diagonal of the tensor product:

$$\mathcal{H}_{\text{diag}} := \text{Span}(|\Psi_{E_L}\rangle \otimes |\Psi_{E_R}\rangle)_{E\in\mathbb{R}^+} \tag{5.22}$$

Which formalizes the notion that only Casimir-equivalent states defined bulk solutions to JT gravity. Analogous to the Hartle-Hawking state, we wish to construct a state on a spacelike slice cutting through the Euclidean wormhole: This corresponds to a minimal geodesic running along the axial direction, connecting β_L and β_R. Normalizing the inner product such that

$$\langle\Psi_{E_L}|\Psi_{E_R}\rangle = \rho(E)\delta(E_L - E_R) \tag{5.23}$$

Where the contribution to the inner product of a given state $|\Psi\rangle \propto e^{-\frac{1}{2}\beta_L\hat{C}\Psi_L} = e^{-\frac{1}{2}\beta_L E_L}$, we recover the thermofield-double state:

$$|\text{TFD}(\beta_L,\beta_R)\rangle = \int_0^\infty dE\; e^{-\frac{1}{2}(\beta_L+\beta_R)E}\,|\Psi_E\rangle_L \otimes |\Psi_E\rangle_R \tag{5.24}$$

Which describes a quantum state on the JT gravity wormhole. To recover the two-boundary Euclidean partition function, we compute the norm of the thermofield double state:

$$\langle \text{TFD}|\text{TFD}\rangle = \int_0^\infty dE\, dE'\, e^{-\frac{1}{2}(\beta_L+\beta_R)(E_L+E_R)}\, \langle \Psi_{E_L}|\Psi_{E_R}\rangle \tag{5.25}$$

Which reduces from condition (5.21) to the wormhole partition function:

$$Z(\beta_L,\beta_R) := \langle \text{TFD}|\text{TFD}\rangle = \int_0^\infty dE\, \rho(E)\, e^{-(\beta_L+\beta_R)E} \tag{5.26}$$

This differs from the product of the individual disk partition functions — hence one has nonfactorization on the wormhole:

$$Z(\beta_L)Z(\beta_R) = \int_0^\infty \int_0^\infty dE\, dE'\, \rho(E)\rho(E')e^{-\beta_L E-\beta_R E'} \tag{5.27}$$

$$\boxed{Z(\beta_1,\beta_2) \neq Z(\beta_1)Z(\beta_1)} \tag{5.28}$$

5.4 Extension to more complex manifolds

The issue of nonfactorization is much more general than as it presents in the wormhole. Recall that for the Euclidean gravitational path integral as given in (5.4), one wishes to compute the saddles over all possible 2-manifolds which support the JT structure. In general, a similar phenomenon is observed on higher genus surfaces, where the added complexity of the bulk not only breaks the symmetry further (as $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{U}(1) \to \mathbb{1}_{\mathfrak{g}}$), but also prevents factorization due to the connectedness of the bulk. When computing the correlation function $\langle Z(\beta_1)Z(\beta_2)\rangle$, for example, the full expression as suggested by

(5.9) is given by a sum over all possible bulk geometries with two conformal boundaries:

$$Z(\beta_1, \beta_2) = \underbrace{\sum_{g=0}^{\infty} \int_{\mathscr{M}_{g,2}} \mathcal{D}g_{\mu\nu}\mathcal{D}\phi\, e^{-S_{\text{JT}}[g_{\mu\nu},\phi]}}_{\textbf{I}} + \underbrace{\sum_{g,h=0,0}^{\infty,\infty} \int_{\mathscr{M}_{g,1}} \int_{\mathscr{M}_{h,1}} \mathcal{D}g^1_{\mu\nu}\mathcal{D}g^2_{\mu\nu}\mathcal{D}\phi^1\mathcal{D}\phi^2\, e^{-S_{\text{JT}}[g^1_{\mu\nu},\phi^1]-S_{\text{JT}}[g^2_{\mu\nu},\phi^2]}}_{\textbf{II}} \tag{5.29}$$

Which accounts for all of the possible bulk configurations terms. Term **I** is the higher-genus expansion of contributions from connected, g-handle bulk geometries, and term **II** is the higher-genus sum of all disconnected bulk geometries, where each has an independent number of handles, hence the sum over $\{g, h\}$. Considering the leading terms in each expression (i.e. the $g = 0$ term and the $(g, h) = (0, 0)$ term) yields the product of two discs from **I**, and the wormhole from **II** respectively:

$$Z(\beta_1, \beta_2) \approx Z_{\text{disc}}(\beta_1)Z_{\text{disc}}(\beta_2) + Z_{\text{wormhole}}(\beta_1, \beta_2) + \cdots \tag{5.30}$$

It is precisely the existence of these connected terms that prevent factorization.

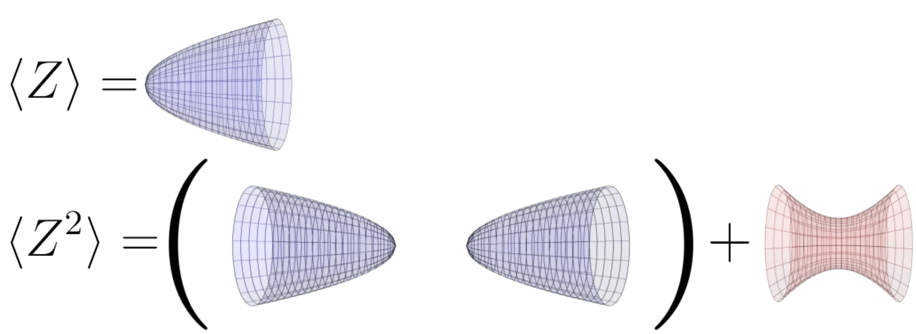

Figure 6: A diagrammatic representation of the terms above, preventing factorization on $Z(\beta_1, .\beta_2)$. (Photo from Santos et. al., AdS Wormholes)

6 Interpretations of Nonfactorization in JT gravity

While the geometric origin of nonfactorization is well understood, its meaning and significance as a challenge to traditional holography remains to some extent a mystery. In the following section, we survey the main interpretations of this fact, which will serve as the motivating tension for the remainder of this work.

6.1 The Wormhole as a Single-Boundary System in JT Gravity

The calculation above forms the primary motivation of one of the overarching themes of bulk quantum gravity theory, mainly that well-defined theories of quantum gravity in the bulk may not require a factorizing boundary Hilbert space as one might expect from traditional holography. This is the perspective of Daniel Harlow and Daniel Jafferis [20]. Their central thesis is that JT gravity, when considered on the wormhole geometry, is both a well-defined theory of quantum gravity (i.e. it has a dual boundary theory and permits bulk states), however it simultaneously lacks a well-defined boundary CFT. While this may not seem significant, it is contrary to the expectation of traditional AdS/CFT. Furthermore, Harlow and Jafferis argue that one should think of JT gravity as — fundamentally — a "single boundary theory". That is, states in the bulk are defined with respect to the inner produce of Hartle-Hawking states such that

$$Z_{JT}(\beta) = \langle \text{HH}|\text{HH}\rangle \tag{6.1}$$

Such that the nonfactorizing state is recovered in the bulk. This is essentially the line of reasoning outlined in Section 5: Because the equations of motion prevent JT gravity from having non-isometric Schwarzian boundaries on boundaries connected by a bulk, not all boundary pairs may realize a possible bulk geometry, and hence the theory should not even be expected to factorize.

From this perspective, a reinterpretation of the single-boundary Euclidean path integral is suggested: Because the pure JT gravity system violates traditional AdS/CFT duality, the subsequent interpretation of $Z(\beta)$ as the thermal partition function may be an oversight. This interpretation also has the effect that wormholes — and by extension any higher-genus surfaces in JT gravity — ought not to contribute to the microstate counting of the black hole entropy, since they do not have well-defined CFT duals. This ontology is derived from an axiomatic perspective, rooted in quantum mechanics: theories which do not factorize violate the corresponding axiom of quantum mechanics, and thus should not be considered when computing observables.

One possible interpretation of the work of Harlow and Jafferis is that the notion that pure JT gravity does not support "multi-boundary" configurations in the traditional sense. As evident from the wormhole solutions, the theory only has the degrees of freedom given on one boundary. Therefore, any state in the bulk JT theory can be described by a single-boundary Hartle-Hawking state.

The analysis of Harlow and Jafferis is done entirely in the Lorentzian signature. They do not consider the off-shell configurations of the bulk to be meaningful, as from a Lorentzian perspective these are nonphysical solutions, despite the fact that they are typically included — though exponentially surpressed — in the Euclidean gravitational path integral. This reinforces the central notion of their work: All solutions beyond the single-boundary, physical solutions are not meaningful in articulating a well-defined CFT.

Hence higher-genus, multi-boundary configurations — as well as off-shell configurations of the theory — due to their violation of factorization, do not contribute meaningfully to pure JT gravity. The exception to this rule sets the stage for the alternate approach to the issue of factorization, which Harlow and Jafferis acknowledge: One may sum arbitrary topologies and off-shell configurations so long as the theory is treated as an ensemble dual. This "ensemble average" dual theory will be a central object of study of the remainder of this thesis, as it has provided a nonperturbative definition of Euclidean JT gravity as dual to an ensemble average of Hamiltonians on the boundary, expressed as a double-scaled matrix integral.

6.2 THE ENSEMBLE-AVERAGE INTERPRETATION

The work of SSS has had a profound impact on the understanding of 2D quantum gravity. Inspired by the emergent AdS_2 symmetry of a random matrix model of quantum gravity — the Sachdev-Ye-Kitadev model [21, 21, 14] — SSS have shown a duality between the perturbative expansion of the Euclidean path integral of JT gravity and a double-scaled matrix integral.[30] This duality provides a nonperturbative expression for the genus ex-

[30]For an introduction to the SYK model and its connections to the work of SSS, see Appendix D

pansion in Euclidean JT gravity, where the genus expansion is precisely equivalent to the "$\frac{1}{L^2}$-expansion" determined by the matrix integral. Furthermore, the duality provided by SSS suggests an "ensemble" factorization of JT gravity, such that the bulk theory is dual to an ensemble of Hamiltonians on the boundary. To contextualize the following chapter which seeks to reinterpret the nonperturbative formalism SSS through the ontology of Harlow and Jafferis, we provide an overview of the key results and techniques developed by SSS.

6.3 Constructing a Factorizing Theory: The Procedure

The central result of SSS was a nonperturbative, factorizing expression for the n-boundary partition function of JT gravity, $Z_{\text{grav}}(\beta_1, ..., \beta_n)$. For the sake of clarity, we begin with the single-boundary expansion, just as in SSS. Recall from the expansion (5.29) that the Euclidean gravitational path integral for a single boundary takes the form

$$Z(\beta_1, \beta_2) = \sum_{g=0}^{\infty} \int_{\mathscr{M}_{g,2}} \mathcal{D}g_{\mu\nu}\mathcal{D}\phi\, e^{-S_{\text{JT}}[g_{\mu\nu},\phi]} \tag{6.2}$$

$$Z(\beta_1, \beta_2) = Z_{\text{disc}}(\beta) + \sum_{g=1}^{\infty} e^{S_0(1-2g)} \int_{\mathscr{M}_{g,1}} \mathcal{D}g_{\mu\nu}\mathcal{D}\phi\, e^{-S_{\text{JT}}[g_{\mu\nu},\phi]} \tag{6.3}$$

Where each term in the expansion corresponds to a bulk with g handles.

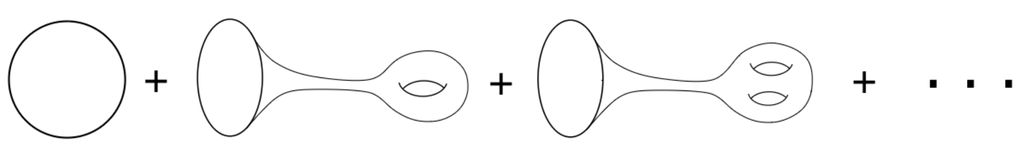

Figure 7: Illustration of the first few topologies in the $(g, 1)$ expansion. (Photo from [1])

The general procedure for computing these higher-genus terms is as follows: One considers "gluing" a trumpet with a conformal boundary of length β to a g-handled bulk, integrating over the moduli space of possible bulk geometries for each Schwarzian mode the trumpet may have. The moduli space of such bulk geometries can be parameterized by the minimal geodesic along the horocycle where the trumpet is glued to the bulk.

For a given geodesic of length b, the space of possible bulk geometries is given by the Weil-Petersson volume, which for a generic manifold takes the form $V_{g,n}(b_1, ..., b_n)$ As given originally by [63], and used in [1]. The $\{b_i\}$ are understood to be the n minimal geodesics onto which the trumpets are glued. Critically, this volume assumes that the bulk metric restricts to the identity on each minimal geodesic. Thus, for a given b, one has the total space of bulk geometries for a given genus. To then consider the whole manifold, one must glue the trumpet geometry along this minimal geodesic such that the metrics align. To do this, however, requires a method of interpolating between the Schwarzian state density $\rho_{\prec}(E)$ on the trumpet's conformal boundary and the minimal geodesic length b.

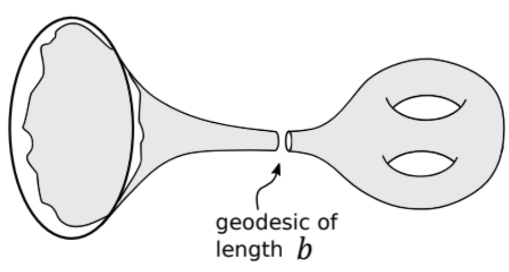

Figure 8: An illustration of the 2-handled, 1-boundary manifold, which has a moduli space dependent on the length of b_i. (Photo from [1])

6.3.1 CONSTRUCTING THE TRUMPET

A key object derived in the SSS model is that of the "half-trumpet", which has one conformal boundary of length β and another fixed to be length b. For a Schwarzian eigenstate density $\rho(E)$, the following prescription gives a method for converting the Casimir into a minimal geodesic along which the trumpet is glued to the bulk. To begin, consider the hyperbolic coordinates from which the trumpet will be cut:

$$ds^2 = dr^2 + \cosh^2(r)d\tau^2, \quad \tau = \tau + b \tag{6.4}$$

Note that this geometry has a residual U(1) axial symmetry. Taking the boundary to be at some fixed length, we have

$$ds^2\Big|_{\text{bdry}} = \cosh^2(r_c)d\tau^2 = \frac{(\partial_u \tau)^2}{\cosh(r_c)^2}du^2 \tag{6.5}$$

Where $\tau(u)$ is an element of Diff(S^1). The induced Schwarzian action on the boundary is then given by a coordinate transformation (as in the case of the hyperbolic disc):

$$S_{\text{bdry}} = -\gamma \int_0^\beta du \, \{e^{-\tau(u)}, u\} \tag{6.6}$$

Which allows for the computation of the partition function.[31] Expanding the Schwarzian modes perturbatively (as in (2.30)) and doing some integration by parts then yields an explicit expression of the trumpet's partition function:

$$Z_{\prec}(b,\beta) = \sqrt{\frac{\gamma}{2\pi\alpha\beta}}e^{-\gamma b^2/2\beta} \tag{6.7}$$

[31] γ is a constant proportional to the eigenstate density.

Which gives an explicit density connecting the density of b to the density of Casimir eigenstates.[32] Recall from expression (5.15) that a broken bulk symmetry implies reparameterizations are only bulk-isometric if they differ by $\mathrm{U}(1)$ symmetry. Hence,

$$|f\rangle_1 \cong_{\mathrm{SL}(2,\mathbb{R})} |f\rangle_2 \not\Rightarrow |f\rangle_1 \cong_{\mathrm{U}(1)} |f\rangle_R \quad \text{for } |f\rangle_i \in \mathrm{Diff}(S^1) \tag{6.8}$$

This is not necessarily a problem, as the holonomy is a topological observable, and is hence insensitive under non-bulk-isometric parameterizations which give the same holonomy. In the following chapter, we will examine this "coarse-graining" in much more detail.

6.3.2 Constructing the Wormhole from Trumpets

Now that we have an expression for the partition function of the trumpet in terms of the minimal attaching geodesic, we can construct the wormhole by gluing two trumpet geometries to a bulk manifold with zero handles and two minimal geodesics, corresponding to the term $V_{0,2}(b_1, b_2)$. This corresponds to the leading term in $Z(\beta_1, \beta_2)$ as expressed in (6.2):

$$Z_{WH}(\beta_1, \beta_2) = \int_{\mathscr{M}_{0,2}} \mathcal{D}g_{\mu\nu}\mathcal{D}\phi e^{-S_{JT}[g_{\mu\nu},\phi]} \tag{6.9}$$

Which, in the language of the above machinery, can be expressed as in integral parameterized by b_1, b_2 performed over the Weil-Petersson volume, together with the trumpet

[32] Note the $\alpha^{-1/2}$ in the final expression. This is a normalization constant from the Gaussian integrals — it will be omitted for clarity, as it can be absorbed into S_0.

partition functions computed above. All together, this gives an expression for the wormhole contribution:

$$Z_{WH}(\beta_1, \beta_2) = \int_0^\infty \int_0^\infty b_1 db_1 b_2 db_2 V_{0,2}(b_1, b_2) Z_{\prec}(b_1, \beta_1) Z_{\prec}(b_2, \beta_2) \tag{6.10}$$

Which can be evaluated explicitly. Note the includion of an additional factor of b_i in each integeal: This follows from the fact that there is a U(1)-redundancy in the gluing of a wormhole to the minimal geodesic: This is the same U(1) bulk symmetry found in (5.15). Hence a geodesic of length b_i allows for a "twist parameter" $(0 \leq \tau < b)$ between the trumpet and the bulk.

As geometric intuition might suggest, the wormhole only has one minimal geodesic; this is the minimal throat of the hyperbolic geometry. As mentioned in Appendix B, the Weil-Petersson volume for the Wormhole is a normalized delta function $\delta(b_1 - b_2)$. This will not happen in general, but it reaffirms the earlier perspective from our discussion of the JT gravity wormhole that no off-shell configurations exist.

Evaluating (6.10), one finds an explicit expression for the wormhole partition function:

$$Z_{WH}(\beta_1, \beta_2) = \int_0^\infty bdb \left(\sqrt{\frac{\gamma}{2\pi\beta_1}} e^{-\gamma b^2/2\beta_1} \right) \left(\sqrt{\frac{\gamma}{2\pi\beta_2}} e^{-\gamma b^2/2\beta_2} \right) = \frac{\sqrt{\beta_1 \beta_2}}{2\pi(\beta_1 + \beta_2)} \tag{6.11}$$

6.4 More Complex Manifolds and the full $\mathcal{Z}_{\text{GRAV}}$

The calculation above is, in fact, very similar for manifolds with an arbitrary number of handles and boundaries. For a specific topology with g handles and n boundaries, one integrates over the Weil-Petersson volume where each $V_{g,n}(b_1,...,b_n)$ is then multiplied by the corresponding array of trumpets given by $Z_{\prec}(\beta_1,b_1),...,Z_{\prec}(\beta_n,b_n)$.

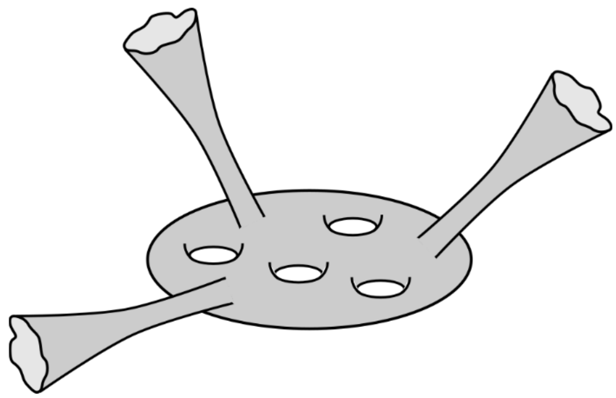

Figure 9: An illustration of the four-handled, three-boundary manifold. The contribution of this piece is given by a triple integral over the minimal geodesics $\{b_1,b_2,b_3\}$ with measure $V_{4,3}(b_1,b_2,b_3)$. (Photo from [1])

The expression for $Z_{g,n}$ with conformal boundaries $\{\beta_i\}$ is the natural extension of equation (6.10) is given as:

$$Z_{g,n}(\beta_1,...,\beta_n) = \int \left[\prod_{i=1}^{n} b_i db_i Z_{\prec}(\beta_i,b_i)\right] V_{g,n}(b_1,...,b_n) \tag{6.12}$$

$$Z_{\prec}(b_i,\beta_i) = \sqrt{\frac{\gamma}{2\pi\beta_i}} e^{-\gamma b_i^2/2\beta_i} \tag{6.13}$$

From which we can write the full partition function, which contains a sum over all possible connected bulk geometries as:

$$\langle \mathcal{Z}(\beta_1,...,\beta_n)\rangle_{\text{conn.}} = \sum_{g=0}^{\infty} \frac{Z_{g,n}(\beta_1,...,\beta_n)}{(e^{S_0})^{2g+n-2}} \tag{6.14}$$

Where the left hand side is manifestly an expectation value, due to the duality to the random matrix expression. Any non-connected geometries can be given as products of these connected pieces, hence the above object characterizes the full perturbative expansion of JT gravity as dual to the double-scaling limit of a random matrix integral.

6.5 The Random Matrix Dual

To make sense of the central thesis of the work of Saad Shenker and Stanford, one first requires an understanding of the "random matrix dual" theory they construct. For a review of the study of random matrix theories, see [64, 65, 66], and for an introduction to their connection to JT gravity and SYK, see Appendix D. For now, let us consider the generic setup of Random matrix theories, outlining their connection to JT gravity as we progress.

Recall from section 4 that the partition function of JT gravity on the disc was given by

$$Z_{0,1}(\beta) = \mathrm{Tr}(e^{-\beta H}) = \int_0^\infty \rho(E)e^{-\beta E}, \quad \rho(E) = \sinh(2\pi\sqrt{E}) \tag{6.15}$$

Which was derived from Casimir density of the principal series representations of of the Schwarzian states. As we have seen, this expression fails to include contributions from the connected topologies which arise in manifolds with more than one boundary. As seen in section 5, the issue of factorization manifests as

$$Z(\beta_1, ..., \beta_n) = Z(\beta_1)...Z(\beta_n) + Z_{\text{wormhole}}(\beta_1, ..) + ... \tag{6.16}$$

Which prevents a unique dual boundary theory from describing the bulk. However, if one considers a particular distribution of Hamiltonians represented by $L \times L$ Hermitian matrices $\{H\}$, then the partition function is no longer an exact expression, but rather an expectation value. Translating the above expression to a matrix integral [1, 66] reads:

$$Z(\beta_1, ..., \beta_n) = \left\langle \prod_{j=1}^{n} \mathrm{Tr}(e^{-\beta_j H}) \right\rangle = \int dH \mu(H) \prod_{j=1}^{n} \mathrm{Tr}(e^{-\beta_j H}) \tag{6.17}$$

$$\mu(H) = e^{-L\mathrm{Tr}(V(H))}, \int dH \mu(H) := 1 \tag{6.18}$$

Where the $L\mathrm{Tr}(V(H))$ encodes the probability of a given matrix H in the ensemble draw, and gives a normalized measure $\mu(H)$ on the distribution. $V(H)$ is a potential defining the theory, analogous to the Boltzmann potential in statistical mechanics. That is, we have an integral over a distribution of Hamiltonians, where for each Hamiltonian, we compute the partition function and average over the ensemble. The important data of the above expression is the eigenvalue density of the ensemble. For any Hermitian matrix H, one has $H \cong U \Lambda U^\dagger$ where $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_L)$, hence one may define a measure on eigenvalues [66] as:

$$dH = [dU] \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_i d\lambda_i \tag{6.19}$$

Where $[dU]$ is the Haar measure on the unitary group $\mathrm{U}(L)$. From this expression, one may rewrite expression (6.17) as

$$Z(\beta) = \langle \mathrm{Tr}(e^{-\beta H}) \rangle \approx \int d\lambda_1 ... d\lambda_L \rho_L(\lambda_1, ..., \lambda_L) \sum_{i=1}^{L} e^{-\beta \lambda_i} \tag{6.20}$$

$$\rho_L(\{\lambda_i\}) = \prod_{i<j} (\lambda_i - \lambda_j)^2 e^{-L \sum_i V(\lambda_i)} \tag{6.21}$$

Where "$\approx$" is suggestive of the fact that at finite L, the distribution ρ is discrete. Since

JT gravity has a continuous eigenvalue spectrum, this can only potentially give an equality in the limit $L \to \infty$ limit. Note that this machinery naturally generalizes to an n-point correlation function. For a partition function with n conformal boundaries, one has:

$$Z(\beta_1, ..., \beta_n) = \left\langle \prod_{j=1}^{n} \mathrm{Tr}(e^{-\beta_j H}) \right\rangle \approx \int d\lambda_1 ... d\lambda_L \rho_L(\lambda_1, ..., \lambda_L) \prod_{j=1}^{n} \left[\sum_{i=1}^{L} e^{-\beta_j \lambda_i} \right] \tag{6.22}$$

Which will provide the contribution for the n-boundary manifolds in $\mathcal{Z}_{\text{grav}}$.

6.5.1 The Double Scaling Limit

Note that the above equation is an approximation, since the distribution of eigenvalues for a matrix of finite rank is discrete. However, for a suitably-steep[33] rational $V(H)$, one recovers an analytic eigenvalue density in the limit, with $\mathcal{O}(1/L)$ fluctuations for finite L.

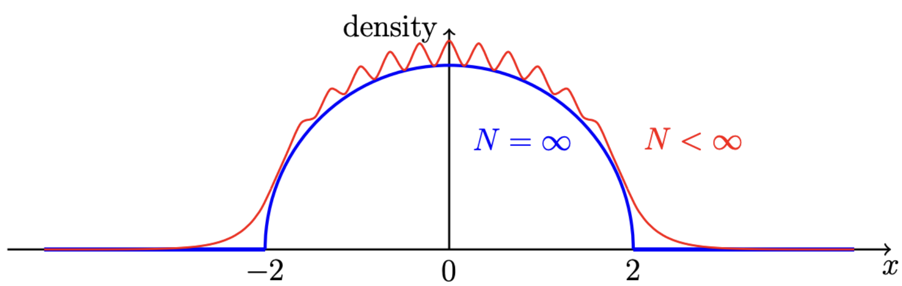

Figure 10: For a Gaussian potential $V(H) \sim H^2$, one recovers the famous Wigner's semicircle law: The eigenvalue distribution is given by a semicircle, where finite L approximations have wavelike fluctuations with frequency $\sim \mathcal{O}(L)$ and amplitude $\mathcal{O}(1/L)$. (Photo from [66])

We denote $\rho(x)$ to be the "equilibrium density" of eigenvalues, explicitly defined as

$$\overline{\rho}(x) = \lim_{L\to\infty} \frac{1}{L} \left\langle \sum_{i=1}^{L} \delta(x - \lambda_i) \right\rangle \tag{6.23}$$

[33]Technically, one requires $\lim_{x\to\infty} V(x) - \log|x^2| = \infty$. For a rigorous derivation, see [67].

Up to now, we have considered ρ as a joint density of a particular set of eigenvalues $\{\lambda_i\}_{1\leq i\leq L}$ — the expression $\rho(x)$ is the coarse-grained density, indifferent to correlations. This is the leading-eigenvalue density — if this random matrix formulation is to be dual to JT gravity, we wish this to correspond to the Plancherel Casimir density $\rho(E)$.

Because JT gravity does not factorize, any successful theory must account for the contribution of $Z_{\text{conn}}(\beta_1, ..., \beta_n)$ to the path integral — in the random matrix model, these contributions will arise as correlation functions in the eigenvalue density. Note that the two-point connected correlation $\rho(x_1, x_2)$ can be expressed similarly in a $L \to \infty$ limit:

$$\rho_L(x_1, x_2) = \left\langle \sum_{i,j=1}^{L} \delta(x_1 - \lambda_i)\delta(x_2 - \lambda_j) \right\rangle - \left\langle \sum_{i=1}^{L} \delta(x_1 - \lambda_i) \right\rangle \left\langle \sum_{j=1}^{L} \delta(x_2 - \lambda_j) \right\rangle \quad (6.24)$$

$$\overline{\rho}(x_1, x_2) = \lim_{L\to\infty} \rho_L(x_1, x_2) \quad (6.25)$$

Which gives the connected two-point correlator. It is no coincidence that formula (6.24) takes the same form of

$$Z(\beta_1, \beta_2) = Z_{\text{conn}}(\beta_1, \beta_2) - Z(\beta_1)Z(\beta_2) \quad (6.26)$$

Though now with expectation brackets.

6.5.2 Ribbon Diagrams and $\mathcal{Z}_{\text{GRAV}}$ Expansion

Now that we have developed the framework of these n-point correlators, it will be fruitful to interpret them not as algebro-geometric objects, but as Feynman diagrams. Recall from QFT that any n-point correlation function of (Gaussian) random variables can be written in terms of a product over propagators Δ_{ij}:

$$\langle x_{ti_1} x_{i_2} \dots x_{i_n} \rangle = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \sum\limits_{\text{pairings}} \prod\limits_{\text{pairs } (i,j)} \Delta_{ij} & \text{if } n \text{ is even.} \end{cases} \tag{6.27}$$

Where Δ descends from $V(H)$, and hence determines the space of correlation functions. Furthermore — just as in QFT — one has that the correlation function $\langle x_{i_1}, ..., x_{i_n} \rangle$ can be expressed diagrammatically as a sum over Feynman diagrams. Similar rules apply: for an expectation of n variables, where each x_{i_k} is repeated p_k times, one has a sum over diagrams, weighted by the corresponding propagators, and dividing by the automorphisms.

These correlators will prove immensely useful in computing the Euclidean path integral of JT gravity, as they provide a recipe for the ensemble calculation of Z in terms of expectation values of products of H. Recall that in equation (6.17), the object which determines $Z(\beta_1, ..., \beta_n)$ is a product of traces of $e^{\beta_j H}$. For the case of the general ensemble given by $V(H)$, the power series of each trace gives a product of matrix elements $\sim \langle H_{ij} H_{kl} ... H_{pq} \rangle$. For a Gaussian propagator, Wick's theorem gives: $\langle H_{ij} H_{kl} \rangle = \frac{1}{L} \delta_{il} \delta_{jk}$.

Continuing the analogy to QFT, recall that graphs are indexed by the number of vertexes. This is similar to the expansion in Random matrix theory, however there is an importance difference: unlike in QFT, RMT diagrams have a nontrivial two-dimensional structure: the edges (i.e. matrix indices) attached to each vertex are ordered cyclically, which define an oriented polygonal face, introducing nontrivial complications to the counting. For a given k, the corresponding graphs which represent $\langle \mathrm{Tr}(H^k) \rangle$ are hence given as "ribbon" diagrams, which are Feynman diagrams with this extra structure.

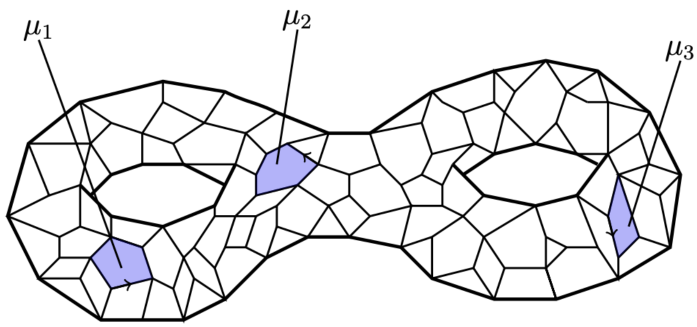

Figure 11: Because ribbon diagrams store nontrivial data on each face, they are often represented as "maps" rather than 1-dimensional graphs. Above is an example of a map with genus 2 and three boundaries, where insertions of H correspond to boundary loops. (Photo from [66])

Note that in ribbon diagrams, edges correspond to contraction: from the above expectation value, this contributes a factor of $1/L$. Faces correspond to free indexes which are summed over, contributing a factor of L. Vertexes count the exponent k, which brings in another factor of L from the coefficient in (6.17). Thus, for a given ribbon diagram, one has a prefactor of L^b given by

$$b = \text{Faces} - \text{Edges} + \text{Vertecies} \tag{6.28}$$

Which is precisely the Euler characteristic χ. This gives $\rho(x_1, ..., x_n)$ as a perturbative expansion in factors of $1/L^2$, corresponding to the factors of g indexed by χ. Thus we have corrections from the leading-order contribution $\overline{\rho}$ which take the form

$$\rho_{\text{conn}}(x_1, ..., x_n) = \lim_{L\to\infty} \sum_{g=0}^{\infty} \frac{\rho^{(g)}(x_1, ..., x_n)}{L^{2g+n-2}} \tag{6.29}$$

Where $\rho^g(x_1, ..., x_n)$ is the amplitude from the ribbon diagrams of genus g. Note that this equation immediately implies $\rho_0(x) = \overline{\rho}(x)$ — the limit distribution of eigenstates is given by the zero-genus (disc) contribution, which aligns with expectations.

We can recover $Z_{g,n}(\beta_1, ..., \beta_n)$ from $\rho(x_1, ..., x_n)$ via a Laplace transform:

$$Z_{g,n}(\beta_1, ..., \beta_n) = \int_0^\infty dx_1 ... dx_n \rho^{(g)}(x_1, ..., x_n) e^{-\sum \beta_j x_j} \tag{6.30}$$

Plugging this into (6.22), and choosing $L = e^{S_0}$ gives:

$$\langle Z(\beta_1, ..., \beta_n)\rangle_{\text{conn}} = \sum_{g=0}^{\infty} \frac{1}{e^{S_0(2g+n-2)}} \int dx_1 ... dx_n \rho^{(g)}(x_1, ..., x_n) e^{-\sum \beta_j x_j} \tag{6.31}$$

$$\langle Z(\beta_1, ..., \beta_n)\rangle_{\text{conn}} = \sum_{g=0}^{\infty} \frac{Z_{g,n}(\beta_1, ..., \beta_n)}{e^{S_0(2g+n-2)}} \tag{6.32}$$

Which matches the previous construction exactly. This is the contribution to $\mathcal{Z}_{\text{grav}}$ from the n-boundary connected component. As we have seen, this is expressed as a perturbative sum of ribbon map coefficients over g, such that the coefficient corresponding to the (g, n) manifold has $L^{-\chi(g,n)}$. The terms $\rho(x_1, ..., x_n)_{\text{conn}}$ (and their genus perturbations) derive from a particular choice of $V(H)$ — in the case of [1], one chooses $V(H)$ such that $\overline{\rho}(x) \propto \sinh(2\pi\sqrt{E})$ such that the Plancherel density is recovered in the $g = 0$ case.

6.6 Interpretations of RMT Wormholes

The ensemble dual (6.32) provided by [1] has raised many questions regarding the scope of traditional AdS/CFT, however it has also shed light on the nonfactorization puzzle itself. The dictionary between the perturbative expansion in JT gravity and the random matrix integral provides a new perspective on the nonfactorization — note that in QFT [34], one has for a two-point correlator:

$$\langle\phi(x)\phi(y)\rangle = \langle\phi(x)\rangle\langle\phi(y)\rangle + \langle\phi(x)\phi(y)\rangle_{\text{conn}} \tag{6.33}$$

Where the last term is the contribution from a nontrivial correlation function. Note that in Lorentzian spacetime, $[\phi(x), \phi(y)] = 0$ if x and y are spacelike separated events. Though it is an interesting phenomenon, it is not a paradox in any sense of the word – the commutator $[\phi(x), \phi(y)]$ still vanishes for spacelike separation, but entanglement persists, because the Wightman (symmetric) correlator

$$\langle 0|\phi(x)\phi(y)|0\rangle \tag{6.34}$$

Remains non-zero, so the vacuum is an entangled state across the two commuting local algebras, and the Hilbert space of states still factorizes. In the context of JT gravity, the canonical example of nonfactorization is first seen in the wormhole, which written in the language of SSS is given by:

$$\langle Z(\beta_1, \beta_2)\rangle = \langle Z(\beta_1)\rangle\langle Z(\beta_2)\rangle + \langle Z_{\text{WH}}(\beta_1, \beta_2)\rangle \tag{6.35}$$

Where each expectation value is understood as dual to the asymptotic genus expansion as in (6.32). The similarity to QFT is by no accident: It is precisely the RMT formulation of JT gravity's genus expansion that suggests the non-factorizing contributions are due to nonzero correlations between boundary states. However, this nonzero contribution from the connected correlator is understood in JT gravity as a geometric bulk connection, which as we have seen prevents a tensor-product decomposition as in QFT. More precisely, one cannot factorize connected bulk geometries since there is no notion of "independent substates" as one has in QFT. Recall that the $F = 0$ constraint in JT gravity prevents independent selection of boundary Schwarzians — this is more fundamental than entanglement: In the Harlowian sense, not only are independent boundary entangled: they are geometrically constrained to be.

The caveat to the above point is if one instead labels each boundary by its own copy of an independent Hilbert space such that the full boundary space is $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, then a connected bulk saddle simply correlates — rather than factorizes — the states on each respective boundary. In this interpretation, the generic Hilbert space is indeed $\bigotimes_i^{\text{\#of universes}} \mathcal{H}_i$, and a connected bulk configuration (g, n) can be understood as an entanglement structure between universes. For a unique bulk (or a finite combination of bulks), the boundary Hilbert space can be understood as a unique, nonfactorizing dual theory, just as one does not have factorization in entangled operator algebras in QFT. The full sum however $g \to \infty$ complicates the perspective: Including the entire sum of bulk geometries, as [1] have shown, requires a dual theory constructed from an ensemble of Hamiltonians.

7 The Ensemble and the Fine-Grained Partition Function

In this section, we explore a "fine-grained" prescription for the construction of bulk geometries in [1]. The procedure outlined above is a prescription for computing the genus expansion through the lens of Random Matrix theory, where the geometric picture gives a gluing procedure for affixing trumpet geometries to a given bulk via an interpolation between the bulk moduli (Weil-Petersson) volumes and the Schwarzian Casimir density. This procedure is, in some sense, a coarse-graining procedure for identifying states by topological invariants corresponding to the observables in BF theory (See chapter 4).

In this chapter, we refine the duality between Jackiw-Teitelboim (JT) gravity and random matrix ensembles by incorporating the full boundary reparameterization data. While the coarse-grained description in terms of Casimir eigenvalues suffices for topological observables, fine-grained correlations among Schwarzian modes exhibit chaotic behavior on their own.

In Saad-Shenker-Stanford [1], each fixed-boundary Casimir eigenvalue corresponds to an ensemble description of the genus expansion, where contributions to the eigenvalue density arise through topological recursion and moduli-weighted bulk geometries. However, the projection from the full space of Schwarzian modes onto Casimir eigenvalues overwrites fine-grained information about distinct Schwarzian states in the trumpet partition function: distinct boundary reparametrizations corresponding to the same Casimir are treated equivalently, potentially losing correlations that survive in individual realizations. The goal of this fine-grained procedure is to better understand the origin and necessity of the ensemble in describing the full behavior of JT gravity.

Though simplistic, the discussion below provides clarity on the distinction between multiple mechanisms of chaos within JT gravity, suggestive that a more refined dual theory may be needed to account for the fine-grained moduli space dynamics.

7.1 The Fine-Grained Construction

To begin, we wish to adapt the SSS gluing procedure to account for the fine-grained data. We begin by acutely identifying the coarse-graining in the SSS model, reformulating it as an integral explicitly containing the fine-grained Schwarzian structure. Recall the gluing procedure outlined in chapter 6 required that the induced metric along the minimal geodesic be trivial,[34] implying that the boundary metric (upon gluing) must be gauge-equivalent to the identity modulo a rescaling corresponding to the holonomy b. By the equations of motion in JT gravity, this implies the conformal boundary parameterization in gluing must also be identity up to $\mathrm{SL}(2,\mathbb{R})$ — we have traded a density of Schwarzian Casimirs for a density of minimal horocycles. While the partition function of both objects has support in $\mathbb{R}^+$, the true distribution of Schwarzian modes is much larger, though only

[34] up to a Fenchel-Nielsen parameter τ to account for the $\mathrm{U}(1)$-invariance

significant for fine-grained statistics. This construction implicitly defines a map

$$\mathrm{P} : \mathcal{S} \xrightarrow{\text{Casimir}} \hat{C}\,|f\rangle \xrightarrow{E=b^2/2\gamma} \{b\} \tag{7.1}$$

$$\mathrm{P} : \mathrm{Diff}^+(S^1)/\mathcal{G}_{\text{bulk}} \longrightarrow \mathbb{R}^+ \tag{7.2}$$

Where the infinite-dimensional boundary data is coarse-grained to the positive real line of Casimirs.[35] The resulting expression for the trumpet partition function therefore captures the contributions of boundary reparameterizations only through their associated holonomy, without resolving finer distinctions among states with identical Casimir. For all purposes of computing observables, this is fine: recall that in the BF theory formulation, observables are contained in topological invariants (i.e. $\mathrm{Hol}(|f\rangle), \mathrm{Tr}(B^2)$), and hence are insensitive to coarse-graining.

The purpose of the analysis below is to investigate the late-time spectral behavior of JT gravity, ideally making a connection between fine-grained statistics to the Spectral Form Factor.

[35] The space $\{|f\rangle\}_{\text{sch}} := \mathcal{S}/\mathcal{G}_{\text{bulk}}$ more precisely is an infinite-dimensional, connected, and simply connected Fréchet manifold with natural symplectic structure arising from the Kirillov-Kostant-Souriau (KKS) construction on the coadjoint orbits of the Virasoro group. The projection thus collapses each of these orbits onto a single b.

7.1.1 THE FINE-GRAINED PARTITION FUNCTION

We now wish to write out the fine-grained partition function, where the structure of the Schwarzian modes is retained. Our starting point will be the expression (6.13), given as:

$$Z_{g,n}(\beta_1, ..., \beta_n) = \int_{\mathcal{M}_{g,n}} \mu_{WP} \prod_{i=1}^{n} \left[\frac{\gamma}{\sqrt{2\pi\alpha\beta}} e^{-\gamma b^2/2\beta} \right] \tag{7.3}$$

Which encodes topologically the flatness constraint on boundary modes,

$$F = 0 \leftrightarrow f_1 \circ \cdots f_n \cong 1 \tag{7.4}$$

Note that this condition is insensitive to coarse-graining, as $F = 0$ is a topological condition and hence only depends on the Casimir of the mode. To formulate this as an explicit integral over all Schwarzian modes, we propose a possible refinement: Rather than consider the integrand to be over the trumpet partition functions, one considers the full space of Schwarzians associated with a given minimal geodesic as the analogous object. Recall that the trumpet partition function was obtained via a Laplace transform on the partition function of Schwarzian Casimirs, and is given as:

$$Z_{\text{sch}}(\beta) = C \int_0^{\infty} \sinh(2\pi\sqrt{E})\, e^{-\beta E} \tag{7.5}$$

To write this as an integral over all Schwarzian modes, the corresponding object should thus be the equivalent integral, but rather than integrating over Casimirs, we integrate over the full space $\mathcal{S}$. This is known as the Alekseev-Shatashvili (AS) action:

$$Z_{\text{bdry}}(b_i, \beta_i) = \int_{\mathcal{S}_{b_i}} \frac{\mathcal{D}\mu_{KKS}(f_i)}{\text{Vol}(\text{SL}(2,\mathbb{R}))}\, e^{-S_{\text{AS}}[f_i;\beta_i]} \tag{7.6}$$

$$S_{AS}[f_i, \beta_i] = C \int_0^\beta du\, \{f(u), u\}, \quad \mathcal{S}_b = \{|f\rangle \in \mathcal{S} \mid \hat{C}\,|f\rangle = E_b := b^2/2\gamma\} \tag{7.7}$$

That is, the domain of the integral is over all Schwarzian modes with fixed Energy. $\mathcal{D}\mu_{KKS}(f_i)$ is the canonical (Kirillov-Konstant-Souriau) symplectic measure, normalized by the coadjoint orbit[36] of $\text{SL}(2,\mathbb{R})$ which leaves the Schwarzian action invariant. Substituting yields:

$$Z^{\text{fine}}_{g,n}(\beta_1, \dots, \beta_n) = \int_{\mathbb{R}^n_+} \prod_{i=1}^n \left(b_i db_i \int_{\mathcal{S}_{b_i}} \mathcal{D}\mu_{KKS}(f_i)\, e^{-S_{\text{AS}}[f_i;\beta_i]} \right) V_{g,n}(b_1, \dots, b_n) \tag{7.8}$$

As the extension of the SSS action to include the fine-grained structure. This partition function can be understood as the equivalent to coarse-grained one with an added structure: At each point $\{b_i\}$ in the coarse grained structure, one now has a fiber bundle of all Schwarzians which satisfy $\hat{C}\,|f\rangle_i = b_i^2/2\gamma$. Explicitly, one has:

$$\mathcal{E}_{b_1,\dots,b_n} = \mathcal{S}_{b_1} \times \cdots \times \mathcal{S}_{b_n}, \quad \mathcal{S}_b := \{f \in S \mid \hat{C}f = b^2\} \tag{7.9}$$

Which is the implicit fiber of Schwarzians above the Saad-Shenker-Stanford construction, embedded in the trumpet partition function. The fiber bundle $\mathcal{E}_{b_1,\dots,b_n}$ is analogous to the space of residual gauge symmetry in BF theory — invariant under topological observables, but potentially significant for late time chaos. The fine-grained structure is hence

[36]The same orbit explored in Section 4 on $\text{SL}(2,\mathbb{R})^+$ BF theory

fully consistent with the coarse-grained SSS gluing at the level of topological sectors, yet retains additional microstructure within each Casimir fiber.

7.2 Recovering the Early Time Spectral Form Factor

Now that we have a partition function for a given bulk, the next step in computing the Spectral Form factor is to perform a summation over genera. As discussed, the Moduli space of bulk geometries is topologically constrained, and thus the genus expansion follows identically for the fine-grained partition function. Adapting the general procedure in Euclidean JT gravity, one has

$$Z_n^{\text{fine}}(\beta_1, ..., \beta_n) = \sum_{g=0}^{\infty} \frac{Z_{g,n}^{fine}(\beta_1, ..., \beta_n)}{(e^{S_0})^{\chi_{g,n}}} \tag{7.10}$$

Which behaves identically when considering topological observables. For the sake of computation, let us consider the $n = 1$ case in constructing the Spectral Form Factor (SFF). The SFF is a Fourier transform on the connected correlator of states; its dip–ramp–plateau profile is a diagnostic of level repulsion and quantum chaos. The generic form of the SFF calculated via analytic continuation of thermal boundary time, expressed as:

$$\text{SFF}(t) = Z(\beta + it)Z(\beta - it)^* \tag{7.11}$$

For the fine-grained partition function above, the SFF for the case $n = 1$ is:

$$\text{SFF}^{\text{fine}}(t) = \sum_{g,h=0}^{\infty} \frac{1}{(e^{S_0})^{2(g+h)-1}} Z_{g,1}^{\text{fine}}(\beta + it) Z_{h,1}^{\text{fine}}(\beta - it)^* \tag{7.12}$$

$$Z_{g,1}^{\text{fine}}(\beta) = \int_{\mathbb{R}^+} bdbV_{g,1}(b) \left[\int_{\mathcal{S}_b} D\mu_{KKS}(f) e^{-S_{AS}[f;\beta]} \right] \tag{7.13}$$

To probe the early-time behavior, we wish to consider the leading-order contribution in the topological expansion. The $g = n = 0$ term has trivial Weil-Petersson Volume, and hence the SFF can be reduced to:

$$\text{SFF}_0^{\text{fine}}(t) \approx e^{2S_0} \int_{\mathcal{S}\times\mathcal{S}} \mathcal{D}\mu(f)\mathcal{D}\mu(\tilde{f})e^{-S_{AS}[f;\beta+it]-S_{AS}[\tilde{f};\beta-it]} \tag{7.14}$$

Furthermore, we consider the typical expansion in Schwarzian modes[37] — valid at early times — such that:

$$f(u) = u + \epsilon(u) + \mathcal{O}(\epsilon^2) \tag{7.15}$$

$$S_{AS}[f, \beta \pm it] = C \oint_0^{\beta\pm it} du \, \{f(u), u\} \approx C \oint_0^{\beta\pm it} du \, \epsilon''(u)^2 \tag{7.16}$$

Which reduces the expression for the SFF down to product of Gaussian integrals

$$\text{SFF}_1(t) \approx e^{2S_0} \int \mathcal{D}\epsilon \, \mathcal{D}\tilde{\epsilon} e^{-\oint \epsilon''(u)\,du - \oint \tilde{\epsilon}''(u)\,du} = e^{2S_0} \oint_{\gamma_1,\gamma_2} \sqrt{\det(\partial_u^4)} \, du \tag{7.17}$$

Where γ_+, γ_- are the contours corresponding to $(0, \beta \pm it)$, and the simplification down to an integral over the determinants follows from performing the functional integration over Gaussian fluctuations in the Schwarzian field. This integral is well known from Virasoro algebras to yield (for each contour) a form $(\beta \pm it)^{-3/2}$, which corresponds to a SFF of

$$\text{SFF}_1(t) \approx \frac{e^{2S_0}}{(\beta+it)^{3/2}(\beta-it)^{3/2}} = e^{2S_0}(\beta^2 + t^2)^{-3/2} \tag{7.18}$$

Which is the first-order approximation of the leading order terms, valid at short timescales $t \ll e^{2S_0}$. As a sanity, check, note that the corresponding SFF calculated from $\rho(E)$ is given

[37] As in section 2.3.2

at leading order as $\rho(E) \sim 2\pi\sqrt{E} + ...$, which yields the power-law decay, corresponding to the early-time decay in the full SFF.

7.3 Chaotic Dynamics of the Schwarzian Fibers

As we have seen, holding topology fixed still suggests that the dynamics within a given fiber of $\mathcal{S}_b$ exhibit genuine quantum chaos. To probe further, let us examine the complexity growth from these Schwarzian fibers in terms of their Lyapunov exponents. In this case, the Schwarzian "Out-of-Time-Correlator" (OTOC) is known to grow exponentially [68, 21] at early times. It is given to be:

$$C_{\text{OTOC}}(t) \sim e^{(2\pi/\beta)\,t}, \tag{7.19}$$

Which is a contribution purely from the fixed-bulk Schwarizan [69], and saturates the Maldacena-Shenker-Stanford bound on chaos [70]. Furthermore, higher modes in the Schwarzian action mix distinct Fourier modes ϵ_n, coupling an initially local excitation across the entire coadjoint orbit on the time-scale $t \sim e^{S_0}$ [21]. Thus the fibers alone are sufficient to scramble information up to the "complexity time" commonly associated with maximal chaos. The boundary Schwarzian fibers thus influence complexity growth for times $t \sim e^{S_0}$, suggesting the asymptotic divergence from higher genus topologies is not the only contribution of chaos for this timescale.

7.3.1 Boundary and Topological Chaos

To grasp more deeply where the late-time chaos originates, it is helpful to consider the full genus expansion. For the case of one boundary, the above formulae reduce to an expression of the form:

$$Z_1^{\text{fine}}(\beta) = \int_{\mathbb{R}^+} \sum_{n=0}^{\infty} \frac{V_{g,1}(b)\, b}{e^{S_0(2g+n-2)}} db \left[\int_{\mathcal{S}_b} D\mu_{KKS}(f) e^{-S_{AS}[f;\beta]} \right] \tag{7.20}$$

Which has the same divergence mentioned in Section 6.6.1, as $V_{g,1}$ still diverges factorially [1], independent of the graining. This should not be a surprise — the asymptotic divergence is a topological property, hence it is insensitive to a choice of graining. The interpretation that each successive topology provides a unique Casimir density spectrum is understood in the fine-grained picture as the asymptotic divergence of the genus-indexed Schwarzian spectra.

In the fine-grained formulation, the onset of chaotic noise at late times can be seen to arise for two reasons. The previously discussed, fixed-genus behavior is suggestive of quantum chaos at times $t \gg e^{S_0}$. The second source of chaos is from the genus expansion, where higher-genus terms are suppressed at first, then diverge for large g. This is the mechanism by which the coarse-grained SSS model reproduce the signature "ramp" and "plateau" of random matrix theory.

Our rudimentary analysis of the fine-grained description is suggestive that chaos in JT gravity arises from two mechanisms. Chaos in the nonperturbative Schwarzian dynamics at the boundary, and graining-independent topological chaos from the factorial divergence of the Moduli volumes, which suggest some necessity in the RMT dual.

7.4 Interpretation of the Ensemble

The above chapter was intended to clarify the ontological status of the ensemble average dual in JT gravity. In this consideration, we have constructed a "fine-grained" adaptation of the SSS partition function, in which we considered the partition function of JT gravity as an integral over all possible microstate configurations, rather than the usual topological reduction.

One interesting facet of this RMT dual is its smoothing of the topological chaos. Since the prescription for constructing bulk geometries fundamentally restricts the Schwarzian fibers, it is difficult to distinguish the effect of the RMT dual on smoothing the chaos of the Schwarzian dynamics on each fiber, or if the topology-fixed chaos examined above at early times is projected out only as a means for computability.

As discussed in section 5, there is no theoretical issue in constructing a unique dual theory for a fixed-topology, though the hypothetical existence of a nonperturbative, unique boundary Hamiltonian dual to the full summand remains unknown. If every consistent completion of the full JT genus expansion necessitates an average over Hamiltonians, then the ensemble description would appear to be the more fundamental, nonperturbative language for two–dimensional quantum gravity.

7.4.1 Quantum Complexity and Fibered Chaos

As we have seen, the two origins of chaos in JT gravity have an interesting interpretation through the lens of quantum complexity, particularly with respect to the Brown-Susskind Conjecture [71, 72], which suggests an upper bound on complexity growth at times $t \sim e^{S_0}$. The fine-grained analysis suggests a narrative for this complexity growth: At early timescales $t \ll e^{S_0}$, it seems complexity is driven by dynamical boundary chaos from the Schwarzian modes, as the higher genera contributions are suppressed by powers of $e^{-\chi S_0}$.

At late times, however, it appears complexity is further driven by topological chaos arising from the divergent genus expansion, which has only found nonperturbative answers via ensemble-averaging techniques — the "ramp" and "plateau" of the late-time SFF are hallmarks of RMT, and bolster the ensemble interpretation as ontologically meaningful beyond its computability. This complexity growth aligns closely with the conjectured duality [73, 74] between complexity of states in quantum gravity and Euclidean wormhole volume — if this conjectural relationship holds in AdS_2, a the complexity of a dominant saddle-point contribution to $\mathcal{V}_{\text{saddle}}$

$$\text{Complexity} \overset{?}{\propto} \frac{\mathcal{V}_{\text{saddle}}}{G_N} \tag{7.21}$$

May be increasingly important at late times.

8 SUMMARY

The aim of this work has been to understand nonfactorization in JT gravity through multiple lenses. We began with the origins of this theory of gravity and its manifestation as a natural theory to describe the near-asymptotic, Robinson-Bertotti geometry of black holes. We then introduced an alternative formulation of this theory as a topological ”BF theory”, noting the structure and duality between bulk and boundary states as a consequence of the topological invariance of the JT gravity's observables. BF theory provided a crisp, topological understanding of the dynamics in JT gravity: it was from this point that we turned our attention to the frontier: The issue of nonfactorization.

Using the language of BF theory, we introduced nonfactorization as the unavoidable consequence of the application of the Euclidean gravitational path integral, examining the canonical spacetime wormhole from which nonfactorization first emerges. The wormhole states were presented in the language of entangled Thermofield Double (TFD) states, noting the connections between Euclidean geometry and Lorentzian entanglement.

This formalism provided us with the language to attack one of the most puzzling questions in JT gravity: the puzzle of nonfactorization. Nonfactorization remains a possible violation to the traditional view of AdS/CFT, and challenges the limits of holography in spacetimes with multiple conformal boundaries. Two primary interpretations of this nonfactorization phenomenon were noted as the representative perspectives from the literature, forming the core analytical landscape of this work. The first, proposed by Harlow and Jafferis, approaches the problem primarily from a Lorentzian perspective.

We discussed their Lorentzian perspective on JT gravity as a well-defined quantum system which fundamentally corresponds to a single boundary theory. From this perspective, the ontological status of boundary states was challenged — since not all pairs of boundary states correspond to a bulk geometry, one ought not consider them as possible realizations of distinct quantum systems. In section 6, we discussed the significance of this condition with respect to typical entanglement, where individual states of an entangled pair can be understood as independent entities.

The latter end of this thesis was devoted to the outlining and interpreting the work of Saad Shenker and Stanford (SSS), who demonstrated a remarkable duality: The full, asymptotic genus expansion in JT gravity is a precise dual to a specific double-scaled matrix integral. As we have seen, this duality reinterprets nonfactorization as correlation functions within an ensemble average of boundary Hamiltonians. This is a bold and profound interpretation of Euclidean JT gravity, yet the full extent to which the "ensemble" describes more fundamental physics present in generic theories of quantum gravity remains uncertain.

This was the central focus of the final chapter, where we probed the individual contributions to the chaos of JT gravity via an analysis of the "fine-grained" partition function. A discussion on the dual mechanisms of chaos in the theory was provided, suggesting dependence on both the fine-grained Schwarzian microstates as well as the topological divergence from which the ensemble-average was derived. As we have studied, the latter topological chaos is not tractable via perturbative techniques. As of now, the full connection bulk and boundary complexity remains uncertain.

8.1 Concluding Remarks

Quantum gravity remains one of the deepest mysteries in theoretical physics. Due to the incredible complexity of the gravitational path integral, many higher-dimensional scenarios remain unsolvable with current techniques, leaving theoretical development scattered across toy models from which broken pieces of insight have arisen. One such model, Jackiw-Teitelboim (JT) gravity, has proved especially powerful in extracting analytic understanding about solvable modes of quantum gravity. Despite its topological nature, Euclidean JT gravity's nonfactorization on connected bulk geometries conlfict with traditional AdS/CFT holography: A bulk spacetime in JT gravity is no longer dual to a tensor-product decomposition on the boundary.

The paradox of nonfactorization in JT gravity has resulted in conceptually strange conclusions, namely the ensemble-averaged, random matrix dual. While conclusions drawn of this duality vary within the literature, the ontological status of the ensemble as a descriptor of JT gravity remains a question of paramount significance toward a more complete understanding of holography and quantum gravity.

A THE PETER-WEYL THEOREM

There is one more important theorem that will become useful later on, and it is the Peter-Weyl theorem.

The Peter-Weyl theorem: If G is a compact, unitary group, then the space of all square-integrable functions on G, $L^2(G)$, can be decomposed into a direct sum of unitary irreducible representations of G. Formally this can be stated as

$$L^2(G) \simeq \bigoplus_{\pi \in \hat{G}} V_\pi^{\oplus \dim V_\pi} \tag{A.1}$$

Where V_π is the representation space of π, and the $\oplus$ counts the multiplicity (degree) of the irrep. Continuing with the connection to quantum mechanics, one can consider the space of wavefunctions ψ for a hydrogen atom. Since our Hamiltonian is rotationally-invariant, we know $\psi \sim f(r)w(\theta, \varphi)$. Let's focus on the angle-dependent function, which lives on a sphere of constant radius r. The group of three-dimensional rotations SO(3) has a natural action on the sphere — it turns out that there is one irrep for each odd dimension. That is, we have

$$f \in L^2(SO(3)) \simeq \bigoplus_{l \in \mathbb{N}} \pi_l(SO(3)) \tag{A.2}$$

$$\pi_l : SO(3) \to GL(\mathbb{C}^{2l+1}) \tag{A.3}$$

Naturally, each $2l + 1$-dimensional representation has $2l + 1$ basis vectors, known as spherical harmonics. Thus,

$$\mathrm{Im}(\pi_l) = \sum_{m=-1}^{m=1} a_m Y_l^m(\theta, \varphi) \tag{A.4}$$

$$f = \sum_{l=0}^{\infty} \sum_{m=-1}^{l} a_{lm} Y_l^m(\theta, \varphi) \tag{A.5}$$

This is the textbook formula for the radial part of the Hydrogen atom wavefunction!

In general, we are only interested in so-called ”class functions”, a subset of the total space of square-integrable functions over a compact, unitary group G. These are functions which are invariant under conjugation of group elements, and arise when G represents a gauge field which does not affect physical quantities under conjugation. Such functions can be represented under a smaller basis: rather than requiring all of the irreps, we only need the conjugation-invariant pieces, which, in the language of linear algebra, is simply the trace of the representation. For a given representation $\pi : G \to GL(V)$, define

$$\chi_\pi(g) = Tr(\pi(g)) \tag{A.6}$$

To be the ”character” of the representation. In addition to satisfying orthogonality with one another, these characters form a basis for all class functions of G:

$$\int_G \chi_\pi(g)\overline{\chi_\rho(g)} = \delta_{\pi,\rho} \quad \text{Orthogonality} \tag{A.7}$$

$$f \in L^2_{class}(G) = \sum_{\pi \in \hat{G}} c_\pi \chi_\pi \tag{A.8}$$

B THE WEIL-PETERSSON VOLUME

A necessary step in computing the contribution of a given topology to the JT gravity Euclidean gravitational path integral is determining the total space of allowed bulk geometric configurations which satisfy the equations of motion. In the case of Euclidean JT gravity, this amounts to defining the moduli space of a class of hyperbolic two-dimensional manifolds with a given number of handles (g) and boundaries (n).

The Weil-Petersson volume is the object used in the SSS construction of the bulk, and is defined on the moduli space of hyperbolic manifolds $(\mathcal{M}_{g,n})$ of a given g-handled, n-boundary topology. This descends from a quotient on the general space of hyperbolic manifolds (Teichmüller space) under redundant bulk isometries (the Mapping class group), however (for the sake of not writing a second thesis in this appendix) we will avoid a derivation of this idea here. For an introduction to these volumes and their applications in physics see [75, 76].

The Weil–Petersson volume is then defined in terms of its similarly-named symplectic form $\omega_{\mathrm{WP}} = d\ell \wedge d\tau_i$, which measures the area (in moduli space) between nearby hyperbolic moduli in terms of their geodesic "length" and "twisting", which are written formally as the Fenchel-Nielsen coordinates. To determine the volume, one must wedge together k copies of ω_{WP}, where $k = 3g - 3 + n$ is half the dimension of the moduli space.

Explicitly, one has

$$\mathrm{Vol}_{\mathrm{WP}}(\mathcal{M}_{g,n}) = \int_{\mathcal{M}_{g,n}} \frac{\omega_{\mathrm{WP}}^{3g-3+n}}{(3g-3+n)!} \tag{B.1}$$

These integrals encode volume data which is found in the integrand of the SSS construction, namely (6.13), however we wish to determine another structure on top of the Weil-Petersson volume. As seen in the construction of the trumpet partition function, one interpolates between the Casimir density of the Schwarzian modes and the "cutoff" length of the minimal attaching geodesic. Hence, as a coefficient $V_{g,n}(b_1, ..., b_n)$, we would like to measure the bulk moduli volume parameterized by the minimal geodesics around each boundary. In particular, when the boundaries are geodesic of fixed lengths $b_1, \ldots, b_n$, we may restrict the moduli volume $\mathcal{M}_{g,n}$ to those moduli which satisfy the above condition. Hence,

$$V_{g,n}(b_1, ..., b_n) = \int_{\mathcal{M}_{g,n}(b_1,..,b_n)} \frac{\omega_{\mathrm{WP}}^{3g-3+n}}{(3g-3+n)!} \tag{B.2}$$

Gives the Weil-Petersson volume useful for our discussion. Note that explicit computations of these volumes is in practice done via a recursive structure discovered by Mirzakhani [63]. In the gravitational path integral, they provide the weight associated to a given bulk topology and boundary conditions. Throughout our treatment of JT gravity, the $V_{g,n}$ appear naturally as the integration measure over moduli space at fixed genus and number of boundaries.

C CONFORMAL FIELD THEORIES

The essence of the study of field theories is the study of symmetry. Symmetry gives rise to equations of motion, conserved currents, and charge. For example, one symmetry which is always assumed is coordinate invariance: This gives rise to the conservation of the stress energy tensor: Under a small perturbation $\delta g_{\mu\nu}$, the action changes as:

$$\delta S = \int d^4x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \tag{C.1}$$

For $\delta S = 0$ under a general coordinate transformation, one requires that $\nabla_\mu T^{\mu\nu} = 0$. There is, however, a stricter requirement we can put on theories which are scale-invariant: Such theories do not depend on lengths, and therefore have another symmetry associated known as Weyl invariance:

$$g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x) \tag{C.2}$$

This symmetry is infinite-dimensional, and corresponds to another equation of motion, $T^\mu_\mu = 0$. Theories which possess both of these symmetries are known as conformally invariant, and Weyl transformations on these theories are known as conformal transformations.

C.1 CONFORMAL TRANSFORMATIONS

How does one classify the space of conformal transformations? Without going into too much detail, we ground intuition the case in $\mathbb{R}^d$, and then move to two-dimensions.

Consider an infinitesimal transformation $x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x)$. This distorts the metric in the following way:

$$\frac{\partial x^{\rho}}{\partial x'^{\mu}} = \delta^{\rho}_{\mu} - \partial_{\mu}\epsilon^{\rho} \tag{C.3}$$

$$\delta g_{\mu\nu} = -(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}) \tag{C.4}$$

If we assume ϵ to be a Weyl (conformal) transformation, then $\delta g_{\mu\nu} \propto g_{\mu\nu}$. Let's call this constant of proportionality ω to represent an infinitesimal conformal transformation.

$$\delta g_{\mu\nu} = -(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}) = \omega(x) g_{\mu\nu} \tag{C.5}$$

Taking the trace of both sides then implies:

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = -\frac{2}{d} g_{\mu\nu}\partial^{\rho}\epsilon_{\rho} \tag{C.6}$$

Which, in $\mathbb{R}^{3+1}$, will derive the Poincaré symmetries associated with boosts and rotations, as well as two less-known symmetries: dilation and special conformal transformations. These symmetries form a group, $SO(4,2)$, which contains the Poincaré group as a subgroup.

C.1.1 The Virasoro Algebra

Two-dimensional Conformal Field Theory is unique in that, if one redefines the coordinates (x_1, x_2) such that

$$\epsilon = \epsilon_1 + i\epsilon_2 \quad \bar{\epsilon} = \epsilon_1 - i\epsilon_2 \tag{C.7}$$

$$z = x_1 - ix_2 \quad \bar{z} = x_1 + ix_2 \tag{C.8}$$

Then the solutions to equation 3.9 are given by:

$$\partial_z \bar{\epsilon}(z, \bar{z}) = \partial_{\bar{z}} \epsilon(z, \bar{z}) = 0 \tag{C.9}$$

Said another way, the space of conformal transformations is simply the space of all holomorphic functions $f(z)$ such that

$$z \to f(z) \quad \bar{z} \to \bar{f}(\bar{z}) \tag{C.10}$$

To understand the behavior of these holomorphic functions, one considers infinitesimal transformations of the form $L_n = -z^{n+1}\partial_z$, which induce a local transformation by $z \to z - z^{n+1}$. Any infinitesimal transformation can be expressed as

$$\sum_n \epsilon_n L_n + \bar{\epsilon}_n \overline{L}_n \tag{C.11}$$

Recall that for a conserved current J^0[38], there is an associated charge Q:

$$Q = \int dx_1 J^0(x, t) \tag{C.12}$$

If we compactify the x_1-direction such that $x_1 \sim x_1 + 2\pi$, Q is conserved along a loop around the cylinder. Returning to holomorphic coordinates, the charge Q can now be expressed as

$$Q = -\frac{1}{2\pi}[\oint_{S^1} dz\ J_z(z, \bar{z}) - \oint_{S^1} d\bar{z}\ J_{\bar{z}}(z, \bar{z})] \tag{C.13}$$

Recall that for an infinitesimal translation, the current $J(z) = T(z)\epsilon(z)$. Using the contour-integral formalism, we now have a new basis for conserved charges. Using the

[38] We always will take x_0 to be time.

L_n basis for ϵ, we can now express a basis of conserved currents

$$J^n(z) = T(z)z^{n+1} \qquad \overline{J}^n(\bar{z}) = \overline{T}(\bar{z})\bar{z}^{n+1} \tag{C.14}$$

Or, in terms of the L_n-generators themselves,

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}T(z), \quad \overline{L}_n = \oint d\bar{z}\, \bar{z}^{n+1}\overline{T}(\bar{z}) \tag{C.15}$$

These generators satisfy the commutation relation

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \tag{C.16}$$

$$[\overline{L}_m, \overline{L}_n] = (n-m)\overline{L}_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \tag{C.17}$$

$$[L_m, \overline{L}_n] = 0 \tag{C.18}$$

The structure these L_n generators form is known as the Virasoro Algebra, which is of central importance to conformal field theory. The formula also reveals something shocking about the algebra these generators form when one quantizes the CFT: They contain a constant term with a proportionality constant $c/12$, the central charge of the theory.

D SYK Model and Emergent AdS_2 symmetry

The Saad–Shenker–Stanford (SSS) matrix integral introduced in Chapter 6 provides a powerful nonperturbative description JT gravity as dual to a "specific double-scaled random matrix integral". To tie this to a physical theory, it is helpful to see a specific realization described by said random matrix integral. The Sachdev–Ye–Kitaev (SYK) model is the candidate theory in question: it is a 0+1-dimensional, strongly interacting theory can be solved explicitly in the large N limit, and reproduces the Schwarzian action discussed previously [14, 21, 68].

The model is surprisingly simple. Let $\psi_i(\tau)_{1\leq i\leq N}$ represent N real Majorana fermions with typical anticommutator relation $\{\psi_i, \psi_j\} = \delta_{ij}$, and specify their interaction by the Hamiltonian:

$$H_{\text{SYK}} = \sum_{1\leq i<j<k<\ell\leq N} J_{ijk\ell}\, \psi_i\psi_j\psi_k\psi_\ell \tag{D.1}$$

$$\overline{J_{ijk\ell}} = 0, \quad \overline{J^2_{ijk\ell}} = \frac{3!\, J^2}{N^3} \tag{D.2}$$

where the bar denotes an average. The SYK model is a disorder-averaged model, so the interactions are drawn independently over a Gaussian ensemble. At finite N, the model is non-integrable, however averaging allows for an explicit calculation of a propagator $G(\tau_1, \tau_2)$ and a "self-energy" term Σ[39]:

$$G(\tau_1, \tau_2) = \frac{1}{N}\sum_i \langle \mathcal{T}\psi_i(\tau_1)\psi_i(\tau_2)\rangle, \quad \Sigma(\tau_1, \tau_2) = J^2 G(\tau_1, \tau_2)^3 \tag{D.3}$$

Which allows for an action functional in terms of these Schwinger functions. This is similar to the action functional of the Ising model, only now we must include Σ as a "self-energy" term to account for the non-Gaussian dynamics which are normally solvable

[39]More specifically, these are Schwinger Functions

explicitly in the former.

$$S_{\text{eff}}[G,\Sigma] = -\frac{N}{2}\log\det\big(\partial_\tau - \Sigma\big) + \frac{N}{2}\int d\tau_1 d\tau_2 \left\{\Sigma G - \tfrac{J^2}{4}G^4\right\} \tag{D.4}$$

$$S_{\text{on-shell}}[G] = -\frac{N}{2}\log\det\left[\partial_\tau - J^2G^3\right] + \frac{3NJ^2}{8}\int d\tau_1 d\tau_2\, G(\tau_1,\tau_2)^4 \tag{D.5}$$

The on-shell solution is particularly illuminating, and can be understood as the large-N limit after integrating out the self-energy term. In this limit, one finds that, in energy scales $\omega \ll J$, the derivative term becomes negligible and the Schwinger–Dyson equations admit a solution given as:

$$G_c(\tau) = \frac{1}{(4\pi)^{1/4}}\frac{\operatorname{sgn}\tau}{J|\tau|^{1/2}} \tag{D.6}$$

Which is conformally invariant under $\text{Diff}^+(S^1)$ symmetry in the IR limit [77, 21]:

$$\tau \mapsto f(\tau) : G_c(\tau_1,\tau_2) \;\longrightarrow\; \big(f'(\tau_1)f'(\tau_2)\big)^{1/2}G_c\big(f(\tau_1)-f(\tau_2)\big). \tag{D.7}$$

However, quantum effects break this infinite-dimensional symmetry down to the $SL(2,\mathbb{R})$ subgroup, leaving Schwarzian modes as the residual goldstone modes, where the action of a Schwarzian reparameterization is identical to the JT gravity action up to constants,

$$S_{\text{Sch}}[f] = -\frac{N\alpha_S}{J}\int d\tau\,\{f(\tau),\tau\} \tag{D.8}$$

Becuase these modes emergy as the residual degrees of freedom only in the IR limit, SYK is often referred to as possessing near-AdS_2 ($NAdS_2$) symmetry. Interestingly, coupling SYK to weak, time-dependent sources produces correlation functions that match bulk geodesic distances in nearly-AdS$_2$ JT gravity, with the Schwarzian providing the gravitational boundary action.

One can also see explicitly how the RMT dual theory averages over the SFF in the context of SYK. The computation of Z_{SYK} requires computing an asymptotic sum over interactions between Majorana fermions, which reproduce the ribbon diagrams outlined in section 6. In the finite N regime, the SFF exhibits erratic behavior at late times t, and only in the $N \to \infty$ limit do the ramp and plateau emerge as hallmarks of a disorder-averaged system.

As N is taken to be large, the expansion takes the form

$$\overline{Z(\beta_1)\cdots Z(\beta_n)} = \sum_{g=0}^{\infty} N^{-2g}\, Z^{\text{ribbon}}_{g,n}(\{\beta_a\};\lambda) \tag{D.9}$$

Where Z^{ribbon} is the contribution from a particular class of diagrams — this drawn on the fact that, as we have seen, ribbon diagrams possess topological invariance based on their number of holes and loops, allowing for a topological expansion over terms. This averaged partition function then reproduces SFF of the ensemble-averaged JT gravity expansion as computed in [1, 78]:

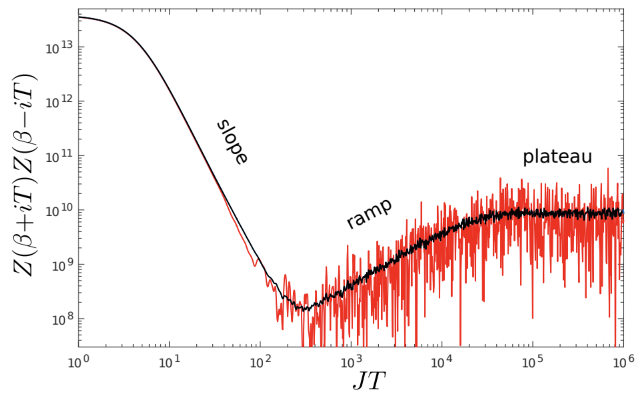

Figure 12: An explicit realization of the SFF as computed in [78] for the case $N = 34$. Note that in the limit of many realizations of J_{ijkl}, one obtains the black line, which is the SFF found in the $N \to \infty$ limit.

Where in the limit $N \to \infty$ one has an equivalence

$$\overline{Z(\beta_1)\cdots Z(\beta_n)} \;=\; Z^{(n)}_{\text{SSS}}(\{\beta_i\}) \tag{D.10}$$

To all orders in the ribbon diagram expansion. From the analysis done by [1, 78], the SYK model has been shown to supply a microscopic quantum mechanical realization of the random-matrix/JT gravity ensemble studied throughout this thesis. Its solvability at large N, maximal chaos, and emergent Schwarzian dynamics suggest a nearly-perfect boundary dual theory of NAdS_2 gravity.

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